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Spherical Functions on a Real Semisimple Lie Group. A Method of Reduction to the Complex Case

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The spherical functions on a real semisimple Lie group (w.r.t. a maximal compact subgroup) are characterized as joint eigenfunctions of certain differential operators on the corresponding complex group. Using this, several results concerning the spherical Fourier transform on the real group are reduced to the corresponding results for the complex group.

When the group in question is a normal real form, this leads to new and simpler proofs of such results as the Plancherel formula, the Paley–Wiener theorem and the characterization of the image under the spherical Fourier transform of the L^1- and L^2-Schwartz spaces. In these proofs neither any knowledge of Harish-Chandras c-function nor the series expansion for the spherical function are used.

For the proof of the main result some analysis of independent interest on pseudo-Riemannian symmetric spaces is developed. Such as a generalized Cartan decomposition and a method of analytic continuation between two "dual" pseudo-Riemannian symmetric spaces.

1. INTRODUCTION

This paper is devoted to the proof and the exploration of Theorem 1.1, which by means of an analytic continuation characterizes the spherical functions on a real semisimple Lie group G_0, by some differential equations on the corresponding complex semisimple Lie group G.

Let g be a complex semisimple Lie algebra. Let g_0 be a noncompact real form of g, with a Cartan decomposition $g_0 = k_0 + p_0$ and Cartan involution σ. Then $u = k_0 + ip_0$ is a compact real form of g, and $k = k_0 + ik_0$ is a complex subalgebra of g,

Now consider g as a Lie algebra over the reals. Let G be a Lie group with Lie algebra g. Let G_0, K, K_0 and U be the analytic subgroups of G corre-

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sponding to g_0, k, k_0 and u, respectively. Let $\mathbf{D}_R(K\backslash G)$ be the set of right invariant differential operators on $K\backslash G$. Let $C^\infty(K\backslash G/U)$ denote the set of C^∞-functions on G, which are left invariant under K and right invariant under U. Notice that K_0 and U are maximal compact subgroups of G_0 and G, respectively, and that K is noncompact.

THEOREM 1.1. *There is a one-to-one correspondence between the set of spherical functions φ on G_0* w.r.t. K_0, *and the set of functions ψ on G, which satisfy the following differential equations and invariance conditions*:

$$\begin{gathered} \psi \in C^\infty(K\backslash G/U), \qquad \psi(e) = 1 \\ D\psi = \lambda_D \psi \qquad \text{for all} \quad D \in \mathbf{D}_R(K\backslash G), \qquad \text{where} \quad \lambda_D \in \mathbb{C}, \end{gathered} \tag{1.1}$$

such that for all $x \in G_0$

$$\varphi(x\sigma(x)^{-1}) = \psi(x).$$

This theorem enables us to "lift" many questions related to analysis w.r.t. the spherical functions on the real group G_0, to analogous questions concerning the spherical functions on the complex group G (w.r.t. U). Since the spherical functions on the complex groups have several simple expressions in terms of elementary functions, many questions are much simpler to handle on these groups. Theorem 1.1 enables us to give new and simpler proofs of several theorems concerning the real groups G_0.

It is then not so much the results we prove for G_0 that we find interesting, as it is the whole structure of the proofs, the new framework in which we can relate the spherical functions on the real groups with the spherical functions on the complex groups, and finally the theory we develop in order to obtain these results.

In the following we briefly outline the content of the paper. In Section 2 we have gathered some notation and a review of some known results concerning spherical functions. In Section 3 we treat in some detail the spherical Fourier transform on a complex semisimple Lie group. We prove the inversion formula, the Plancherel formula and the Paley–Wiener theorem. The main purpose of this section is to demonstrate that the complex groups are easy to work with, and that it therefore makes sense to reduce from the general case to that case.

In the course of the proof we run into a simple formula for the volume of a semisimple compact Lie group U, with maximal torus T:

$$\operatorname{vol}(U) = (2\pi)^{(1/2)\dim U\backslash T} \Big(\prod_{\alpha \in \Delta^+} \langle \alpha, \rho \rangle^{-1} \Big) \cdot \operatorname{vol}(T).$$

This formula has been found independently by Fegan and McDonald [3].

In Section 4 we develop some analysis concerning pseudo-Riemannian symmetric spaces G/H, where G is an arbitrary real semisimple Lie group and H

is a noncompact fixpoint group of an involution. We establish a Cartan decomposition for G/H (Theorem 4.1), which was mentioned without proof in [4]. We use the Cartan decomposition to elaborate on a duality, introduced by Berger [1], between two such pseudo-Riemannian symmetric spaces G/H and G^0/K^0. The results obtained have independent interest with regard to pseudo-Riemannian symmetric spaces. We use them in the beginning of Section 5 to derive Theorem 1.1 above, which is just the special case, where $K\backslash G$ and $G_0 \times G_0/d(G_0)$ play the role of the two dual pseudo-Riemannian symmetric spaces, ($d(G_0)$ is the diagonal subgroup).

If ψ satisfies (1.1), then

$$\Phi(x) = \int_U \psi(ux)\, dx \tag{1.2}$$

is a spherical function on G w.r.t. U. This together with Theorem 1.1 establishes a mapping from the set of spherical functions on G_0 into the set of spherical functions on G. We identify this mapping in terms of Harish-Chandras parameters for the spherical functions (Theorem 5.5). Using this mapping we can find the spherical Fourier transform on G_0, as a "part" of the spherical Fourier transform on G (Theorem 6.1).

We then prove the Paley–Wiener theorem for G_0, provided that G_0 satisfies a certain condition (6.8). This condition is obviously satisfied, when G_0 is a normal form, and we prove it for G_0/K_0 of rank one. This proof of the Paley–Wiener theorem does not use any knowledge of the series expansion of the spherical functions, nor any knowledge of the Harish-Chandra c-function. It is a simple reduction to the complex case.

Our approach works particularly well when G_0 is a normal real form of G. In Sections 7 and 8 we restrict our attention to the normal forms. We prove the inversion formula and the Plancherel theorem (Theorem 7.3), still without using the series expansion of the spherical functions, nor any knowledge of the c-function. Our formula for the Plancherel measure for G_0/K_0 takes the form:

$$d\mu(\lambda) = 2^{(1/2)\dim \mathfrak{a}}\, \pi_0(\rho_0)^{-2}\, \pi_0(\lambda)^2 \int_K \Phi_{2\lambda}(h)\, dh \cdot d\lambda, \qquad \lambda \in \mathfrak{a}_0^*, \tag{1.3}$$

where π_0 is the product of the positive roots and Φ_Λ, $\Lambda \in \mathfrak{a}^*$, are the spherical functions on G. Since Harish-Chandra has shown that $d\mu(\lambda) = |\, c(\lambda)|^{-2}\, d\lambda$, (1.3) is a new formula for the c-function. We obtain a formula dual to (1.2):

$$\varphi_\lambda(x\sigma(x)^{-1}) = 2^{(1/2)\dim \mathfrak{a}}\, \pi_0(\rho_0)^{-2}\, |\, c(\lambda)|^2\, \pi_0(\lambda)^2 \int_K \Phi_{2\lambda}(hx)\, dh, \qquad \lambda \in \mathfrak{a}_0^*, \tag{1.4}$$

for the spherical functions on G_0.

We also give simple proofs of the theorems, which characterize the spherical Fourier transform of the L^1- and L^2-Schwartz-spaces. The only known proofs of these theorems are fairly involved and difficult.

Section 8 is used to prove that the integration over K in (1.3) is allowed. For any specific group it is very easy to prove, by just looking at the root structure. To give a proof, which is independent of classification, is a little more technical and requires an inductive procedure, which essentially is due to Schmid [25] and developed further by Vogan [28]. The proof we give was kindly suggested to us by D. Vogan.

In Section 9 we make a few comments on the representation theoretic aspects of our method. In Section 10 we consider the special case of $\mathbf{SL}(2, \mathbb{R})$.

2. Notation and a Brief Review of the Theory of Spherical Functions

Notation

Except for Section 4, where a separate notation is used, the notation will be as follows:

g is a complex semisimple Lie algebra, g_0 is a noncompact real form of g, with Cartan decomposition $g_0 = k_0 + p_0$ and Cartan involution σ. $u = k_0 + ip_0$ is a compact real form of g, and $k = k_0 + ik_0$ is a complex subalgeba of g. Let $g = u + a + n$ be an Iwasawa decomposition of g, then $g_0 = k_0 + a_0 + n_0$, where $a_0 = a \cap g_0$ and $n_0 = n \cap g_0$, is an Iwasawa decomposition of g_0. Write the Cartan decomposition of g: $g = u + p$, where $p = iu = p_0 + ik_0$.

The real, resp. complex, dual of a is denoted by a^*, resp. $a^*_{\mathbb{C}}$, similarly for a_0. As for the Killing form one should be aware that the Killing form B_0 on g_0 is just the restriction of the complex Killing form B' of g, where as the Killing form B of g, as a real Lie algebra, is $2B'$. This means that the Euclidean structures on a_0, induced by B_0 and B, are different. We shall write $\langle\ ,\ \rangle_0$ and $\|\ \|_0$ for the scalar product and norm induced by B_0 on a_0, and $\langle\ ,\ \rangle$, $\|\ \|$ for the scalar product and norm induced by B on p. So in particular

$$\| H \|_0^2 = \tfrac{1}{2} \| H \|^2 \qquad \text{for all} \quad H \in a_0 . \tag{2.1}$$

Let a_k denote the orthogonal complement of a_0 in a. Then we shall embed a_0^* in a^* by extending $\lambda \in a_0^*$ to be zero on a_k. Similarly we also embed a^* in p^*, the dual of p. If $\lambda \in p^*$ we denote by X_λ the element in p, determined by $\langle \lambda, X \rangle = \langle X_\lambda , X \rangle$ for all $X \in p$. If $\lambda \in a^*$ we often write H_λ instead of X_λ. Let $H_\lambda{}^0$ be the corresponding notion for $(a_0^*, \langle\ ,\ \rangle_0)$. a_0^* and p^* have Euclidean structures induced by these dualities. In particular

$$\tfrac{1}{2} H_\lambda{}^0 = H_\lambda \qquad \text{and} \qquad \| \lambda \|_0^2 = 2 \| \lambda \|^2 \qquad \text{for all} \quad \lambda \in a_0^* . \tag{2.2}$$

Extend $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_0$ to complex bilinear mappings on $p^*_{\mathbb{C}}$, resp. $(a_0)^*_{\mathbb{C}}$.

Let Δ be the root system of the pair (g, a), i.e., Δ is the restricted roots for the real Lie algebra g, such that in particular each rootspace g^α, $\alpha \in \Delta$, has dimension $m_\alpha = 2$. Let Δ_0 be the root system of the pair (g_0, a_0). Then $\Delta_0 = \{\alpha|_{a_0} \mid a \in \Delta, \alpha|_{a_0} \neq 0\}$. Also $\Delta_k = \{\alpha|_{a_k} \mid a \in \Delta, \alpha(a_0) = \{0\}\}$ is a root system on a_k. Corresponding to the Iwasawa decomposition of g, there are natural choices of positive Weyl chambers a^+ and $a_0{}^+$, and positive root systems Δ^+, $\Delta_0{}^+$ and $\Delta_k{}^+$. We define

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha = \sum_{\alpha \in \Delta^+} \alpha, \qquad \rho_0 = \tfrac{1}{2} \sum_{\alpha \in \Delta_0{}^+} m_\alpha \alpha \qquad \text{and} \qquad \rho_k = \tfrac{1}{2} \sum_{\alpha \in \Delta_k{}^+} \alpha.$$

Then

$$\rho = 2(\rho_0 + \rho_k).$$

Let W, W_0 and W_k be the Weyl groups of Δ, Δ_0 and Δ_k. Let $W' = \{s \in W \mid s(a_0) = a_0\}$, then ([29, vol. I. Prop. 1.1.3.3]) we have

$$W_0 = W'/W_k. \tag{2.3}$$

Define $a^{*+} = \{\lambda \in a^* \mid H_\lambda \in a^+\}$ and $a_0^{*+} = \{\lambda \in a_0^* \mid H_\lambda{}^0 \in a_0{}^+\}$.

Let G be a Lie group with Lie algebra g, and let K, U, A, N, G_0, K_0, A_0, and N_0 be the analytic subgroups corresponding to k, u, a, n, g_0, k_0, a_0, and n_0. By the Iwasawa decomposition the mapping $(k, a, n) \to$ kan is a diffeomorphism of $U \times A \times N$ onto G and of $K_0 \times A_0 \times N_0$ onto G_0. Let for $x \in G$, $H(x) \in a$ be determined by $x \in U \exp(H(x))\, N$. (If $x \in G_0$, then $H(x) \in a_0$). Let $\overline{A_0{}^+} = \exp(\overline{a_0{}^+})$. The Cartan decomposition

$$G_0 = K_0 \overline{A_0{}^+} K_0 \tag{2.4}$$

asserts that for each $x \in G_0$, there exists a unique $a \in \overline{A_0{}^+}$ such that $x \in K_0 a K_0$. It can be shown (using [13, X, Theorem 6.10]) that the set $C^\infty(K_0\backslash G_0/K_0)$ of K_0-biinvariant C^∞-functions on G_0, via restriction to A_0, is in bijective correspondence with $C^\infty_{W_0}(A_0)$, the set of W_0-invariant C^∞-functions on A_0. Similarly with $G = U\overline{A^+}U$.

$\mathbf{D}(G)$, $\mathbf{D}(G_0)$, $\mathbf{D}(G/U)$, $\mathbf{D}(G_0/K_0)$ etc. shall denote the algebras of left invariant differential operators on the respective groups or homogeneous spaces. $\mathbf{D}_R(G)$, $\mathbf{D}_R(K\backslash G)$ etc. are the corresponding notions for right invariance. $\mathbf{Z}(G) = \mathbf{D}(G) \cap \mathbf{D}_R(G)$ and $\mathbf{Z}(G_0) = \mathbf{D}(G_0) \cap \mathbf{D}_R(G_0)$ are the centers of $\mathbf{D}(G)$ and $\mathbf{D}(G_0)$, respectively.

As a general reference to the structure of semisimple Lie groups and Lie algebras, we refer to Helgason's book [13]. In particular we refer to Chapter X for the basic properties of spherical functions.

Normalization of measures

It is important to keep straight the normalization of the measures. Here we give a complete list.

1. On compact groups the total mass is one:

$$\int_{K_0} dk = 1, \qquad \int_U du = 1.$$

2. The measures on p_0, p_0^*, a_0 and a_0^* are normalized such that the Euclidean Fourier transform is an isometry. The metric being induced by he Killing form. (I.e., the measure is given by the volume element times $(2\pi)^{-(1/2)d}$ where d is the dimension of the space.)

3. Let π_0, δ_0, and J_0 be the functions defined for $H \in a_0$ by

$$\pi_0(H) = \prod_{\alpha\in\Delta_0^+} (\langle\alpha, H\rangle)^{m_\alpha}, \qquad \delta_0(H) = 2^{l_0} \prod_{\alpha\in\Delta_0^+} (\sinh\langle\alpha, H\rangle)^{m_\alpha},$$
$$J_0(H) = 2^{-l_0}\delta_0(H)\,\pi_0(H)^{-1}, \tag{2.5}$$

where $l_0 = \sum_{\alpha\in\Delta_0^+} m_\alpha = \dim n_0 = \dim K_0/M_0$. J_0 is the Jacobian of the exponential map $\mathrm{Exp}\colon p_0 \to G_0/K_0$, and extends to an $\mathrm{Ad}(K_0)$ invariant strictly positive function on p_0. M_0 is the centralizer of A_0 in K_0. Let m_0 be the Lie algebra of M_0. The metric on K_0/M_0 is induced by $-B_0$ on $m_0^\perp$. We shall need the following integral formulas for $f \in C_c(p_0)$ and $f \in C_c(p_0^*)$

$$\int_{p_0} f(X)\,dX = \gamma_0 \int_{a_0^+}\int_{K_0} f(\mathrm{Ad}(k)\,H)\,\pi_0(H)\,dk\,dH \tag{2.6}$$

and

$$\int_{p_0^*} f(\lambda)\,d\lambda = \gamma_0 \int_{a_0^{*+}}\int_{K_0} f(\mathrm{Ad}^*(k)\,\lambda)\,\pi_0(\lambda)\,dk\,d\lambda \tag{2.7}$$

where $\pi_0(\lambda) = \pi_0(H_\lambda{}^0)$, and $\gamma_0 = (2\pi)^{-(1/2)l_0}\,\mathrm{vol}(K_0/M_0)$.

4. The Riemannian metric on G_0/K_0 is induced by the Killing form on p_0. The measure is the volume element normalized by $2^{l_0}\gamma_0^{-1}$, such that

$$\int_{G_0/K_0} f(x)\,dx = 2^{l_0}\gamma_0^{-1}\int_{p_0} f(\mathrm{Exp}\,X)\,J_0(X)\,dX$$
$$= \int_{a_0^+}\int_{K_0} f(k\,\mathrm{Exp}\,H)\,\delta_0(H)\,dk\,dH, \tag{2.8}$$

for $f \in C_c(G_0/K_0)$.

5. The measures on G_0 and on $G_0 \times G_0/d(G_0)$ are normalized such that

$$\int_{G_0} f(x)\,dx = \int_{G_0/K_0}\int_{K_0} f(xk)\,dk\,dx = \int_{G_0\times G_0/d(G_0)} f(xy^{-1})\,d(x, y) \tag{2.9}$$

for $f \in C_c(G_0)$.

6. Everything above applies, of course, to G, U, $\mathfrak{p}$, $\mathfrak{a}$, etc. instead of G_0, K_0, $\mathfrak{p}_0$, $\mathfrak{a}_0$, etc. π, δ, J, γ, l are defined correspondingly.

By Theorem 4.2 and Lemma 5.1 the measure on $K\backslash G$ can be normalized such that

$$\int_{G_0 \times G_0/d(G_0)} f(x)\, dx = \int_{K\backslash G} f^{\eta}(x)\, dx \tag{2.10}$$

for $f \in C_c(K_0 \times K_0\backslash G_0 \times G_0/d(G_0))$.

Finally the measure on K is normalized such that for $f \in C_c(G)$

$$\int_G f(x)\, dx = \int_{K\backslash G} \int_K f(hx)\, dh\, dx. \tag{2.11}$$

Spherical Functions

Recall that a spherical function φ on G_0 w.r.t. K_0 is defined by:

$$\begin{gathered} \varphi \in C^{\infty}(K_0\backslash G_0/K_0), \qquad \varphi(e) = 1 \\ D\varphi = \lambda_D \varphi \qquad \text{for all} \quad D \in \mathbf{D}(G_0/K_0), \qquad \text{where} \quad \lambda_D \in \mathbb{C}. \end{gathered} \tag{2.12}$$

Harish-Chandra has characterized the spherical functions by the following integral formula:

$$\varphi_\lambda(x) = \int_{K_0} e^{\langle i\lambda - \rho_0, H(xk)\rangle}\, dk, \qquad x \in G_0. \tag{2.13}$$

Here $\lambda \in (\mathfrak{a}_0)_{\mathbb{C}}^*$, and $\varphi_\lambda = \varphi_\mu$ if and only if λ and μ are conjugate under W_0. It is known that $\lambda \to \lambda_D$ (determined by (2.13) and (2.12)) for each D is a polynomial function on $(\mathfrak{a}_0)_{\mathbb{C}}^*$, and that

$$\lambda_\omega = -(\langle \lambda, \lambda\rangle_0 + \langle \rho_0, \rho_0\rangle_0) \tag{2.14}$$

for the Cassimir operator ω.

Let $G_0{}^\wedge$ denote the set of equivalence classes of irreducible, unitary representations of G_0. Let $G_0{}^\wedge(K_0)$ denote the subset of class-one representations w.r.t. K_0, i.e., $\pi \in G_0{}^\wedge(K_0)$ if and only if there exits a vector $v_\pi \neq 0$ (unique up to a scalar), in the corresponding Hilbert space, such that $\pi(K_0)$ leaves v_π fixed. There is a one-to-one correspondence $\pi \leftrightarrow \varphi$, between $G_0{}^\wedge(K_0)$ and the set of positive definite spherical functions, such that $\varphi(x) = \| v_\pi \|^{-2} \times (v_\pi, \pi(x)\, v_\pi)$ for all $x \in G_0$.

The regular representation of G_0 on $L^2(G_0/K_0)$ decomposes as a direct integral over $G_0{}^\wedge(K_0)$, with constant multiplicity one. The corresponding measure $d\mu$ on $G_0{}^\wedge(K_0)$ is called the Plancherel measure for G_0/K_0. In order to determine $d\mu$, it is enough to decompose $L^2(K_0\backslash G_0/K_0)$ according to the positive definite spherical functions. This has been done by Harish-Chandra in [9–11]. We describe briefly his results:

In [9] he gave a series expansion of φ_λ with leading coefficients $c(s\lambda)$, $s \in W_0$. The function $c(\lambda)$ was explicitly determined by Gindikin and Karpelevič [8], as a meromorphic function on $(a_0)^*_{\mathbb{C}}$. Let now for $x \in G_0$, $|x|_0$ be defined as the K_0-biinvariant function, satisfying $|\exp H|_0 = \|H\|_0$ for $H \in a_0$. Define the spherical L^2-Schwartz-space $\mathscr{I}^2(G_0)$ by:

$$\mathscr{I}^2(G_0) = \{f \in C^\infty(K_0\backslash G_0/K_0) \mid \forall N \in \mathbb{N}, D \in \mathbf{D}(G_0): \sup_{x \in G_0} (1 + |x|_0)^N \mid Df(x)| \varphi_0(x)^{-1} < +\infty\}. \tag{2.15}$$

Let $\mathscr{S}_{W_0}(a_0^*)$ be the subspace of the usual Schwartz-space on a_0^* of rapidly decreasing functions, consisting of W_0-invariant functions.

Define the spherical Fourier transform $f \to f^\sim$ by:

$$f^\sim(\lambda) = \int_{G_0} f(x)\, \varphi_{-\lambda}(x)\, dx, \tag{2.16}$$

for $\lambda \in a_0^*$ and $f \in \mathscr{I}^2(G_0)$, (or when ever well defined).

THEOREM 2.1 (Harish-Chandra). (i) *The spherical Fourier transform is a bijection of $\mathscr{I}^2(G_0)$ onto $\mathscr{S}_{W_0}(a_0^*)$. If $\psi \in \mathscr{S}_{W_0}(a_0^*)$ then the inverse spherical Fourier transform f is given by*

$$f(x) = \int_{a_0^{*+}} \psi(\lambda)\, \varphi_\lambda(x) \mid c(\lambda)|^{-2}\, d\lambda, \qquad \textit{for} \quad x \in G_0. \tag{2.17}$$

(ii) *The spherical Fourier transform extends by continuity to an isometry of $L^2(K_0\backslash G_0/K_0)$ onto $L^2(a_0^{*+}, |c(\lambda)|^{-2}\, d\lambda)$.*

A shorter proof of part (ii) of this theorem has been given by Helgason–Gangolli–Rosenberg avoiding the most difficult part of Harish-Chandras proof, i.e., the inductive procedure in [10]. Elaborating on the expansion for the spherical function, but still relying on [9] Helgason and Gangolli in [15], [6] (see also [16, pp. 37–39]) proved the following Paley–Wiener theorem:

THEOREM 2.2. (i) *The spherical Fourier transform is a bijection of $C_c^\infty(K_0\backslash G_0/K_0)$ onto the space $\mathscr{H}(a_0^*)$ of functions in $\mathscr{S}_{W_0}(a_0^*)$, which are the restrictions to a_0^* of entire functions of exponential type on $(a_0)^*_{\mathbb{C}}$.*

(ii) *Furthermore $f \in C_c^\infty(K_0\backslash G_0/K_0)$ has support in a ball of radius R if and only if $f^\sim$ extended to $(a_0)^*_{\mathbb{C}}$ has exponential type R.*

Finally Rosenberg [23] noticed, that the proof of Helgason–Gangolli can be twisted so that it does not rely on Harish-Chandra [10]. Instead one proves (2.17) directly for functions in $C_c^\infty(K_0\backslash G_0/K_0)$.

It should be noted that part (i) of Theorem 2.1 does not follow in this way. This part of the theorem has been generalized by Helgason [16] and Trombi–Varadarajan [27] to the spherical L^p-Schwartz-spaces $0 < p \leqslant 2$:

$$\mathscr{I}^p(G_0) = \{f \in C^\infty(K_0\backslash G_0/K_0) \mid \forall N \in \mathbb{N}, D \in \mathbf{D}(G_0): \sup_{x \in G_0} (1 + |x|_0)^N |Df(x)|^p \varphi_0(x)^{-2} < +\infty\}. \tag{2.18}$$

Here we just comment on the case $p = 1$. Let C_{ρ_0} be the convex hull of the set $\{s\rho_0 \mid s \in W_0\}$ in a_0^*. Helgason–Johnson [18] has shown that

THEOREM 2.3. *φ_λ is bounded if and only if $\lambda \in a_0^* + iC_{\rho_0}$.*

From this follows easily (see [16, p. 28–31]) that if $f \in \mathscr{I}^1(G_0)$, then $f^\sim(\lambda)$ is well defined for $\lambda \in a_0^* + iC_{\rho_0}$, holomorphic in the interior and rapidly decreasing in the appropriate sense. These conditions are also sufficient:

THEOREM 2.4. *The spherical Fourier transform is a bijection of $\mathscr{I}^1(G_0)$ onto the space $\mathscr{H}_{\rho_0}(a_0^*)$ of W_0-invariant, rapidly decreasing holomorphic functions in the tube $a_0^* + iC_{\rho_0}$.*

For G_0 of real rank one or for G_0 complex this was proved by Helgason [16]. In general it was proved by Trombi–Varadarajan [27], by a very complicated inductive procedure, generalizing Harish-Chandra's method in [10].

Finally we should point out that everything said concerning spherical functions on G_0 w.r.t. K_0, also covers the pair G, U (which is just the special case where g_0 has a complex structure). In order to distinguish between the spherical Fourier transform on G and on G_0, we shall write Φ_Λ, $\Lambda \in a_{\mathbb{C}}^*$ for the spherical functions on G, and $F \to F^\wedge$ for the spherical Fourier transform on G. Again we point out, that all the above mentioned theorems are much simpler to prove for G than for G_0 in general.

3. SPHERICAL FUNCTIONS ON THE COMPLEX GROUP G

In this section we shall gather some features, which are specific for the comcomplex case, and we shall prove the inversion formula, the Plancherel theorem, the Paley–Wiener theorem, and a slightly weaker form of the L^2-Schwartz-space theorem.

The spherical functions on G w.r.t. U has the following two formulas, besides (2.13), (see [17, pp. 31–32]):

$$\Phi_\Lambda(\exp H) = \frac{\pi'(\rho)}{\pi'(i\Lambda)} \delta'(H)^{-1} \sum_{s \in W} \det(s)\, e^{\langle is\Lambda, H\rangle} \tag{3.1}$$

for $\Lambda \in a_{\mathbf{C}}^*$ and $H \in a$, (when there is a singularity the formula should be interpreted by continuity).

$$\pi'(\Lambda) = \prod_{\alpha \in \Delta^+} \langle \alpha, H_\Lambda \rangle \qquad \text{and} \qquad \delta'(H) = \prod_{\alpha \in \Delta^+} (2 \sinh\langle \alpha, H \rangle),$$

(i.e., $(\pi')^2 = \pi$ and $(\delta')^2 = \delta$, since $m_\alpha = 2$ for all $a \in \Delta$).

Since $\Phi_\Lambda(\exp H)$ is W-invariant in both Λ and H, we can extend it to a $\mathrm{Ad}^*(U) \times \mathrm{Ad}(U)$ invariant function on $p^* \times p$, as such it is given by the following formula

$$\Phi_\Lambda(\exp X) = J(X)^{-1/2} \int_U e^{\langle i\Lambda, \mathrm{Ad}(u)X \rangle}\, du. \tag{3.2}$$

THEOREM 3.1. (i) *Let $F \in C_c^\infty(U\backslash G/U)$ and $x \in G$, then*

$$F(x) = \int_{a^{*+}} F^\wedge(\Lambda)\, \Phi_\Lambda(x)\, \pi(\rho)^{-1}\, \pi(\Lambda)\, d\Lambda \tag{3.3}$$

(ii) *The spherical Fourier transform is a bijection of $C_c^\infty(U\backslash G/U)$ onto $\mathscr{H}(a^*)$.*

(iii) *It extends to an isometry of $L^2(U\backslash G/U)$ onto $L^2(a^{*+}, \pi(\rho)^{-1}\, \pi(\Lambda)\, d\Lambda)$.*

Proof. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ denote the Euclidean Fourier transform on p and a, respectively. From (2.8) and (3.2) it follows easily that

$$F^\wedge(\Lambda) = 2^l \gamma^{-1} \mathscr{F}_1(F \cdot J^{1/2})(\Lambda), \qquad \Lambda \in p_{\mathbf{C}}^*. \tag{3.4}$$

Now the Euclidean inversion formula, combined with (2.7) and (3.2), gives for $X \in p$:

$$F(\exp X) = 2^{-l} \gamma^2 \int_{a^{*+}} F^\wedge(\Lambda)\, \Phi_\Lambda(\exp X)\, \pi(\Lambda)\, d\Lambda. \tag{3.5}$$

From (2.8) and (3.1) it follows, using $\delta'(sH) = \det(s)\, \delta'(H)$ and $\pi'(s\Lambda) = \det(s)\, \pi'(\Lambda)$, that

$$F^\wedge(\Lambda) = \frac{\pi'(\rho)}{\pi'(-i\Lambda)} \mathscr{F}_2(F \cdot \delta')(\Lambda), \qquad \Lambda \in a_{\mathbf{C}}^*, \tag{3.6}$$

and by the Euclidean inversion formula combined with (3.1), that for $H \in a$:

$$F(\exp H) = \int_{a^{*+}} F^\wedge(\Lambda)\, \Phi_\Lambda(\exp H) \frac{\pi'(i\Lambda)\, \pi'(-i\Lambda)}{\pi'(\rho)^2}\, d\Lambda. \tag{3.7}$$

This proves (i).

Comparing (3.5) and (3.7) we find $\gamma = 2^{(1/2)l}(\pi'(\rho))^{-1}$.

Now let $\mathscr{H}(p^*)$ be the space of $\mathrm{Ad}^*(U)$ invariant, entire, rapidly decreasing functions of exponential type on $p_{\mathbf{C}}^*$. Then the Euclidean Paley–Wiener theorem and (3.4) shows that $F \to F^\wedge$ is a bijection of $C_c^\infty(U\backslash G/U)$ onto $\mathscr{H}(p^*)$. Since the restriction of a function in $\mathscr{H}(p^*)$ to $a_*^{\mathbb{C}}$ belongs to $\mathscr{H}(a^*)$, we have proved that $F^\wedge \in \mathscr{H}(a^*)$, if $F \in C_c^\infty(U\backslash G/U)$. Now assume $\psi \in \mathscr{H}(a^*)$. Define F by (3.3), then by (3.6) and the Paley–Wiener theorem for $\mathscr{F}_2$ it follows that

$$H \to F(\exp H)\,\delta'(H) \quad \text{is in} \quad C_c^\infty(a). \tag{3.8}$$

It follows from [13, X, Theorem 6.10] that ψ extends to a unique $\mathrm{Ad}^*(U)$-invariant Schwartz-function on p^*. Then (3.4) tells us that $F \cdot J^{1/2}$ is a Schwartz-function on p. In particular is $X \to F(\exp X)$ in $C^\infty(p)$. Combining this with (3.8) we find that the restriction of F to A is in C_c^∞, since $\{H \in a \mid \delta'(H) \neq 0\}$ is open and dense. This implies that $F \in C_c^\infty(U\backslash G/U)$, which finishes (ii).

The L^2-isomorphism now follows from (i) and (ii) in the usual way, or by the L^2-theory for $\mathscr{F}_1$ and (3.4). Q.E.D.

Remark. In the course of the proof we have seen that $\gamma = 2^{(1/2)l}\pi'(\rho)^{-1}$. Recall that $\gamma = (2\pi)^{-(1/2)l}\,\mathrm{vol}_B(U/T)$. Where T is the centralizer of a in U, thus T is a maximal torus. The volume is taken with respect to the Killing form B on $g = u + iu$, considered as a real Lie algebra. Thus the Killing form on u is $\frac{1}{2}B$. If we use the volume induced by that, we find since $l = \dim U/T$,

$$\mathrm{vol}(U) = (2\pi)^{(1/2)\dim U/T} \prod_{\alpha\in\Delta^+} \langle\alpha, \rho\rangle^{-1}\,\mathrm{vol}(T). \tag{3.9}$$

(Here $\langle\alpha, \rho\rangle$ has the same value, whether we take $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} 2\alpha$ and $\langle\ ,\ \rangle$ w.r.t. B, or we take $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and $\langle\ ,\ \rangle$ w.r.t. $\frac{1}{2}B$, i.e., whether we consider g as a real or a complex Lie algebra.) See Freudentahl and Vries [5, p. 202] and Warner [30] for similar formulas.

A couple of remarks should be made here. The c-function for G is given by $c(\Lambda) = \pi'(\rho)/\pi'(i\Lambda)$, such that (3.3) is a special case of (2.17).

Another way to prove (ii) of Theorem 3.1, would be to prove directly that any $\psi \in \mathscr{H}(a^*)$ of exponential type R has a unique extension to a function in $\mathscr{H}(p^*)$, also of exponential type R. This can be done along the following lines:

Let $P_1, \ldots, P_n$ be a set of algebraically independent homogeneous generators of $\mathbf{I}_U(p^*)$, the U-invariant polynomials on p^*. The restrictions to a^*, $\bar{P}_1, \ldots, \bar{P}_n$ generate $\mathbf{I}_W(a^*)$ (see [13, Chap. X]). Write $\psi(\Lambda) = \sum_\nu a_\nu \bar{P}_1^{\nu_1}(\Lambda) \cdots \bar{P}_n^{\nu_n}(\Lambda)$, $\Lambda \in a_{\mathbb{C}}^*$. Define ψ on $p_{\mathbb{C}}^*$ by

$$\psi(\Lambda) = \sum_\nu a_\nu P_1^{\nu_1}(\Lambda) \cdots P_n^{\nu_n}(\Lambda), \qquad \Lambda \in p_{\mathbb{C}}^*.$$

Since P_i on p^* is $\mathrm{Ad}^*(U)$ invariant, it follows that P_i on $p_{\mathbb{C}}^*$ is $\mathrm{Ad}^*(U)_{\mathbb{C}}$ invariant. But the action of $\mathrm{Ad}^*(U)_{\mathbb{C}}$ on $p_{\mathbb{C}}^*$, is the same as the action of G on g. Now the result follows from Lemma 3.2, which we state without proof.

LEMMA 3.2. *Let for $X \in g$, $X = X_1 + iX_2$, where $X_1, X_2 \in u$. For any $X \in g$, there exits an $H = H_1 + iH_2 \in a + ia$ in the closure of the G orbit of X, such that $\| H_1 \| = \inf\{\|(\mathrm{Ad}(x)\, X)_1\| \mid x \in G\}$.*

We have not proved Theorem 2.1(i) for the complex group G. It seems to be necessary with Harish-Chandra's inductive procedure for that. This is however easier for the complex groups, where (3.1) is essentially the series expansion of Φ_Λ. The following slightly weaker theorem is an immediate consequence of the fact that the Schwartz-space is preserved by the Euclidean Fourier transform $\mathscr{F}_1$, formula (3.4) and the fact ([17, p. 32]) that for $F \in C_c^\infty(U\backslash G/U)$

$$(\omega F)(\exp X) = J(X)^{-1/2} L(F \cdot J^{1/2})(X), \qquad X \in p,$$

where L is the Laplacian on p. (Recall that $\Phi_0 = J^{-1/2}$.)

THEOREM 3.3. *The spherical Fourier transform on G is a bijection of*

$$\mathscr{I}_0^2(G) = \{F \in C^\infty(U\backslash G/U) \mid \forall N, M \in \mathbb{N} \cup \{0\}$$
$$\times \sup_{x \in G} (1 + |x|)^N \, |\omega^M F(x)| \, \Phi_0^{-1}(x) < +\infty\} \tag{3.10}$$

onto $\mathscr{S}_W(a^*)$.

But we do not know any a priori proof that $\mathscr{I}_0^2(G) = \mathscr{I}^2(G)$. Finally we remark that Helgason's proof of the corresponding result for $\mathscr{I}^1(G)$ (Theorem 2.4) is a simple reduction to the $\mathscr{I}^2(G)$ result.

4. Analysis on Pseudo-Riemannian Symmetric Spaces

Let, in this section only, g denote an arbitrary semisimple Lie algebra over $\mathbb{R}$ of noncompact type. Let τ be any involutive automorphism of g. Then [1] there exists a Cartan involution σ of g, such that $\tau\sigma = \sigma\tau$. Let $g = h + q$, resp. $g = k + p$, be the decomposition of g into $+1$ and -1 eigenspaces for τ, resp. σ. Since τ and σ commute, g decomposes into the direct sum of vectorspaces:

$$g = h \cap k + q \cap k + h \cap p + q \cap p. \tag{4.1}$$

Let g_0 be the subalgebra fixed under the involution $\sigma\tau$, i.e., $g_0 = h \cap k + q \cap p$. Clearly g_0 and h are reductive Lie algebras, with Cartan involution given by the restriction of σ.

Let G be a connected Lie group with Lie algebra g, and finite center. Suppose that τ extends to an involution of G, (σ always extends). Let G_0, H and K be the analytic subgroups of G corresponding to g_0, h and k. Notice that K is a maximal compact subgroup of G. If g_0 is not semisimple, then let $g_0' = [g_0, g_0]$ denote the semisimple part and c_0 the center. Let a_0 be maximal Abelian in $g_0' \cap p$, then $a_0 = a_0' + c_0 \cap p$ is maximal Abelian in $g_0 \cap p = q \cap p$. Now choose a positive Weyl chamber $a_0'^+$ in a_0', and define $a_0^+ = a_0'^+ + c_0 \cap p$. Let W_0 be the Weyl group of (G_0, a_0), define $A_0 = \exp(a_0)$ and $A_0^+ = \exp(a_0^+)$. Notice that by (4.3) below, it is easily seen that $K \cap H$ is connected. We can now extend the Cartan decomposition (2.4) to the reductive group G_0 to give

$$G_0 = (K \cap H)\,\overline{A_0^+}(K \cap H). \tag{4.2}$$

As with (2.4) $C^\infty(K \cap H \backslash G_0 / K \cap H)$ can be identified with $C^\infty_{W_0}(A_0)$.

The homogeneous space G/H becomes a pseudo-Riemannian symmetric space, with the metric induced by the Killing form on q. We shall need the following generalized Cartan decomposition.

THEOREM 4.1. (i) $G = K\overline{A_0^+}H$, *or more precisely: For every* $x \in G$ *there exists a unique* $a \in \overline{A_0^+}$ *such that* $x \in KaH$.

(ii) *The set* $C^\infty(K\backslash G/H)$ *is in bijective correspondence, via restriction to* A_0, *with the set* $C^\infty_{W_0}(A_0)$, *of* W_0 *invariant* C^∞*-functions on* A_0.

Proof. $h \cap p$ is a Lie triple system contained in p, i.e., $[X, [Y, Z]] \in h \cap p$ for all $X, Y, Z \in h \cap p$. By a theorem of Mostow [22], (see also [13, VI, Theorem 1.4] or [21, p. 161]), the mapping

$$(k, X, Y) \to k \exp X \exp Y \tag{4.3}$$

is a homeomorphism of $K \times q \cap p \times h \cap p$ onto G. A simple argument (see [21, pp. 160–161]) shows that this is actually a diffeomorphism. So in particular for each $x \in G$ there is a unique $X \in q \cap p$, such that

$$x \in K \exp X \exp(h \cap p). \tag{4.4}$$

Using (4.2) there is a unique $a \in \overline{A_0^+}$, such that

$$\exp(X) \in (K \cap H)\, a(K \cap H). \tag{4.5}$$

It follows that $x \in KaH$. Now let $a_1 \in \overline{A_0^+}$ be arbitrary such that $x \in Ka_1H$. Let $k \in K$ and $h \in H$ be such that $x = ka_1h$. By (4.3) we can write $h = h_1 \exp Y$, where $h_1 \in K \cap H$ and $Y \in h \cap p$. Thus, $x = kh_1(h_1^{-1}a_1h_1) \exp Y$, and by the uniqueness in (4.3) we have that $h_1^{-1}a_1h_1 = \exp X$, and then, by the uniqueness in (4.5), we have that $a_1 = a$. So (i) is proved. To prove (ii) let $f \in C^\infty_{W_0}(A_0)$. First extend f to an $\mathrm{Ad}(K \cap H)$-invariant function $X \to f(\exp X)$ on $q \cap p$ by (4.2), then use (4.3). Q.E.D.

Remark. If H is a nonconnected subgroup with Lie algebra h of the fixpoint group for τ, then the theorem clearly remains true, provided G_0 is defined by

$G_0 = K \cap H \exp(q \cap p)$, and $\overline{a_0{}^+}$ is a fundamental domain for the action of W_0.

We now turn to describe the duality. Let $g_{\mathbb{C}} = g + ig$ be the complexification of g. Extend σ and τ to complex linear involutions of $g_{\mathbb{C}}$. Define the real subalgebras of $g_{\mathbb{C}}$, dual to g, h, and k as follows:

$$\begin{aligned} g^0 &= g_0 + i(q \cap k + h \cap p), \\ h^0 &= h_{\mathbb{C}} \cap g^0 = h \cap k + i(h \cap p), \\ k^0 &= k_{\mathbb{C}} \cap g^0 = h \cap k + i(q \cap k). \end{aligned}$$

Let σ^0 and τ^0 denote the restrictions of σ and τ to g^0. Let $p^0 = p_{\mathbb{C}} \cap g^0 = ih \cap p + q \cap p$ and $q^0 = q_{\mathbb{C}} \cap g^0 = iq \cap k + q \cap p$. Now notice that $(g^0, \sigma^0, \tau^0, h^0, k^0)$ satisfies the same conditions that we had for (g, τ, σ, k, h) in the beginning of this section. In particular h^0 is a maximal compact subalgebra of g^0.

Notice that $g_0 = h \cap k + q \cap p = h^0 \cap k^0 + q^0 \cap p^0$ corresponds to itself under the duality, such that a_0, $a_0{}^+$ and W_0 are the same for both settings. Now let $G_{\mathbb{C}}$ be any connected linear group over $\mathbb{C}$ with Lie algebra $g_{\mathbb{C}}$, for example $G_{\mathbb{C}} = \mathrm{Int}(g_{\mathbb{C}})$. Let G, G^0, H, H^0, etc. be the analytic subgroups corresponding to g, g^0, h, h_0, etc. Since $C^\infty_{W_0}(A_0)$ is the same for both settings, Theorem 4.1(ii) gives an isomorphism of $C^\infty(K\backslash G/H)$ onto $C^\infty(K^0\backslash G^0/H^0)$. This isomorphism turns out to preserve a lot of properties.

In $C_c^\infty(K\backslash G/H)$, the index c shall mean compact support modulo H, similarly in $C_c^\infty(K^0\backslash G^0/H^0)$ it shall mean compact support modulo K^0. On G/H, resp. on $K^0\backslash G^0$, we use the left-, resp. right-, invariant measure derived from the pseudo-Riemannian structure. $L^p(K\backslash G/H)$ and $L^p(K^0\backslash G^0/H^0)$ then has good meaning.

THEOREM 4.2. (i) *Let $\mathscr{F}$ stand for C, C_c, C^∞, C_c^∞. There is an isomorphism $f \to f^\eta$ of $\mathscr{F}(K\backslash G/H)$ onto $\mathscr{F}(K^0\backslash G^0/H^0)$ uniquely determined by the condition that*

$$f(x) = f^\eta(x) \qquad \text{for all } x \in G_0. \tag{4.6}$$

(ii) *If the measure on $K^0\backslash G^0$ is multiplied by a suitable positive constant, then $f \to f^\eta$ extends to an isometry of $L^p(K\backslash G/H)$ onto $L^p(K^0\backslash G^0/H^0)$ for $1 \leqslant p \leqslant \infty$.*

The only thing left to prove is part (ii). Since the proof of that uses the next theorem, we shall first state and prove that.

Let $\mathbf{I}(p)$, resp. $\mathbf{I}(p^0)$, $\mathbf{I}(q)$, and $\mathbf{I}(q^0)$, denote the algebras of $\mathrm{Ad}(K)$-, resp. $\mathrm{Ad}(K^0)$-, $\mathrm{Ad}(H)$-, and $\mathrm{Ad}(H^0)$-, invariant polynomials on p, resp. p^0, q, and q^0. By [13, X, Theorem 2.7] the symmetrization mapping defines a linear one-to-one mapping of $\mathbf{I}(p)$ onto $\mathbf{D}_R(K\backslash G)$. It is clear that any $Q \in \mathbf{I}(p)$ has a unique extension, called $Q_{\mathbb{C}}$, to $\mathbf{I}(p_{\mathbb{C}})$, i.e. to a $K_{\mathbb{C}}$-invariant polynomial on the complex vector space $p_{\mathbb{C}}$, and that $Q \to Q_{\mathbb{C}}$ is an isomorphism of $\mathbf{I}(p)$ onto $\mathbf{I}(p_{\mathbb{C}})$.

Similar remarks hold for p^0, q, and q^0. Now since $p_{\mathbb{C}} = p_{\mathbb{C}}{}^0$ and $K_{\mathbb{C}} = K_{\mathbb{C}}{}^0$, we have hereby defined a bijective linear mapping $D \to D^\eta$ of $\mathbf{D}_R(K\backslash G)$ onto $\mathbf{D}_R(K^0\backslash G^0)$, and similarly of $\mathbf{D}(G/H)$ onto $\mathbf{D}(G^0/H^0)$.

$\mathbf{I}(p_{\mathbb{C}})$ is in one-to-one correspondence with $\mathbf{D}_R{}^{\mathbb{C}}(K_{\mathbb{C}}\backslash G_{\mathbb{C}})$, the richt invariant holomorphic differential operators on $K_{\mathbb{C}}\backslash G_{\mathbb{C}}$, "holomorphic" understood in the obvious way.

THEOREM 4.3. (i) *For any $f \in C^\infty(K\backslash G/H)$ and any $D \in \mathbf{D}_R(K\backslash G) \cup \mathbf{D}(G/H)$ the following holds*

$$(Df)^\eta = D^\eta f^\eta. \tag{4.7}$$

(ii) *$D \to D^\eta$ is an algebra isomorphism of $\mathbf{D}_R(K\backslash G)$ onto $\mathbf{D}_R(K^0\backslash G^0)$, resp. of $\mathbf{D}(G/H)$ onto $\mathbf{D}(G^0/H^0)$.*

Proof. (i) It is enough to prove for $D \in \mathbf{D}_R(K\backslash G)$ that

$$(Df)^\eta(a) = (D^\eta f^\eta)(a) \tag{4.8}$$

for each $a \in A_0$ and $f \in C^\infty(K\backslash G/H)$, such that f is analytic in a neighborhood of a in G. So let such a and f be fixed. Let Ω be a neighborhood of a in $G_{\mathbb{C}}$, such that f extends to a holomorphic function $f^{\mathbb{C}}$ in Ω. It is clear by analytic continuation that $f^{\mathbb{C}}$ and f^η are identical on $\Omega \cap G^0$. Now let $Q \in \mathbf{I}(p)$, $Q^{\mathbb{C}} \in \mathbf{I}(p_{\mathbb{C}})$ and $Q^0 \in \mathbf{I}(p^0)$ correspond as above. Let $D \in \mathbf{D}_R(K\backslash G)$, $D^{\mathbb{C}} \in \mathbf{D}_R{}^{\mathbb{C}}(K_{\mathbb{C}}\backslash G_{\mathbb{C}})$ and $D^\eta \in \mathbf{D}_R(K^0\backslash G^0)$ be the associated differential operators. Then it follows that:

$$Df(x) = D^{\mathbb{C}} f^{\mathbb{C}}(x) \qquad \text{for all } x \in \Omega \cap G, \tag{4.9}$$

and

$$D^\eta f^\eta(x) = D^{\mathbb{C}} f^{\mathbb{C}}(x) \qquad \text{for all } x \in \Omega \cap G^0. \tag{4.10}$$

Since $a \in \Omega \cap G \cap G^0$, (4.8) follows.

Now in order to prove (ii) take $D_1, D_2 \in \mathbf{D}_R(K\backslash G)$, and define $D = (D_1D_2)^\eta - D_1{}^\eta D_2{}^\eta \in \mathbf{D}_R(K^0\backslash G^0)$. Since D is right invariant it is enough to prove that D is zero at the point $e = \{K^0\} \in K^0\backslash G^0$. So let $g \in C^\infty(K^0\backslash G^0)$ be analytic in a neighborhood of e. Proceeding as under (i) we can find a neighborhood Ω of e in $G_{\mathbb{C}}$, and a holomorphic function $f^{\mathbb{C}}$ in Ω, such that the restriction of $f^{\mathbb{C}}$ to $\Omega \cap G^0$ is g, and the restriction f of $f^{\mathbb{C}}$ to $\Omega \cap G$ is in $C^\infty(K\backslash G)$. Then we find similarly to (4.9) and (4.10) that

$$Dg(e) = (D_1D_2)^{\mathbb{C}} f^{\mathbb{C}}(e) - D_1{}^{\mathbb{C}}(D_2{}^{\mathbb{C}} f^{\mathbb{C}})(e) = D_1D_2 f(e) - D_1(D_2 f)(e) = 0.$$

Q.E.D.

Remark. Since we know that $\mathbf{D}(G/H^0)$ is commutative, G/H^0 being a Riemannian symmetric space (see [13, X, Theorem 2.9]), it follows from (ii) that $\mathbf{D}(G/H)$ is commutative for any semisimple (i.e., G is semisimple) pseudo-

Riemannian symmetric space G/H. The above proof of this is essentially due to Helgason (private communication). (See [20] for a more general result.)

Proof of Theorem 4.2(ii). Let ω, resp. ω^0, be the Laplace–Beltrami operator on G/H, resp. $K^0\backslash G^0$. We can now use the "polar" decomposition of the Laplace–Beltrami operator in Helgason [17, p. 15 (see remark on p. 18)], with $V := G/H$, $H := K$ and $W := A_0^+$, to find that there exists a C^∞-function β on A_0^+ s.t. for all $f \in C_c^\infty(K\backslash G/H)$

$$\int_{G/H} f(x)\,dx = \int_{A_0^+} f(a)\,\beta(a)\,da, \tag{4.11}$$

and for all $a \in A_0^+$

$$\omega f(a) = \beta^{-1/2}(a)\,L(\beta^{1/2}f)(a) - \beta^{-1/2}(a)\,L(\beta^{1/2})(a). \tag{4.12}$$

L is the Laplace operator on A_0. Similarly there is a C^∞-function β_0 on A_0^+, such that for all $f \in C_c^\infty(K^0\backslash G^0/H^0)$

$$\int_{K^0\backslash G^0} f(x)\,dx = \int_{A_0^+} f(a)\,\beta_0(a)\,da, \tag{4.13}$$

and for all $a \in A_0^+$

$$\omega^0 f(a) = \beta_0^{-1/2}(a)\,L(\beta_0^{1/2}f)(a) - \beta_0^{-1/2}(a)\,L(\beta_0^{1/2})(a). \tag{4.14}$$

Clearly $\omega^0 = \omega^\eta$, since ω and ω^0 are the restrictions of the Cassimir operator on G and G^0, respectively. But this implies by (4.12) and (4.14) that β and β_0 are proportional, which by (4.11) and (4.13) is what we should prove.

Q.E.D.

5. Spherical Functions on a Real Semisimple Lie Group G_0 as Functions on the Complex Group G

Now we return to the notation of Section 2. So g_0 is a noncompact real form of g. $g_0 = k_0 + p_0$ is a Cartan decomposition and σ the Cartan involution of g_0. Denote also by σ the extension of σ to a conjugate linear automorphism of g. Let τ be the complex linear extension of σ to g. The fixpoint set for σ is $u = k_0 + ip_0$, for τ is $k = k_0 + ik_0$, and for $\sigma\tau$ is g_0.

Now we can use the results of Section 4, where the role of $(g, h, k, g_0, \tau, \sigma)$ now shall be played by $(g, k, u, g_0, \tau, \sigma)$. The decomposition (4.1) is just $g = k_0 + ip_0 + ik_0 + p_0$. The pseudo-Riemannian symmetric space in question is $K\backslash G$. Fortunately the notation is consistent, except for (h, k) which is now (k, u). So in particular the meaning of a_0 and a_0^+ should be clear.

LEMMA 5.1. *Up to an isomorphism the dual objects g^0, k^0, u^0, σ^0 and τ^0 is given by*

$$g^0 = g_0 \times g_0 ,$$
$$k^0 = k_0 \times k_0 ,$$
$$u^0 = d(g_0) = \{(X, X) \mid X \in g_0\},$$
$$\sigma^0(X, Y) = (Y, X) \qquad \textit{for all } (X, Y) \in g_0 \times g_0 ,$$

and
$$\tau^0(X, Y) = (\sigma(X), \sigma(Y)).$$

The embedding of g_0 in g^0 is given by the mapping $X \to (X, \sigma(X))$. So in particular a_0, as a subalgebra of g^0, is given by

$$\{(H, -H) \mid H \in a_0\}.$$

Proof. This is a straightforward computation using the following complex linear isomorphism Θ between $g_{\mathbb{C}}$ and $g \times g$:

$$\Theta(X, Y) = \tfrac{1}{2}(X - jiX) + \tfrac{1}{2}(\sigma(Y) + ji\sigma(Y)), \tag{5.1}$$

for $(X, Y) \in g \times g$. Where $j = (-1)^{1/2}$ in $g_{\mathbb{C}}$, and i is multiplication by $(-1)^{1/2}$ in g, here considered as a vectorspace isomorphism of the real space g. Q.E.D.

We see from the lemma that the dual pseudo-Riemannian symmetric space G^0/U^0 is just $G_0 \times G_0/d(G_0)$. This is diffeomorphic to G_0 via the mapping

$$(x, y)\, d(G_0) \to xy^{-1}. \tag{5.2}$$

Under this diffeomorphism $\mathscr{F}(K_0 \times K_0\backslash G_0 \times G_0/d(G_0))$ is identified with $\mathscr{F}(K_0\backslash G_0/K_0)$ for $\mathscr{F} = C, C_c, C_c^{\infty}, L^p$, etc. $\mathbf{D}_R(K_0 \times K_0\backslash G_0 \times G_0)$ is identified with $\mathbf{D}_R(K_0\backslash G_0) \otimes \mathbf{D}(G_0/K_0)$, and $\mathbf{D}(G_0 \times G_0/d(G_0))$ is identified with $\mathbf{Z}(G_0)$. Thus we can restate Theorems 4.2 and 4.3 in our situation.

THEOREM 5.2. (i) *Let $\mathscr{F}$ stand for C, C_c, C_c^{∞}, C^{∞} or L^p, $1 \leqslant p \leqslant \infty$. There is an isomorphism $f \to f^{\eta}$ of $\mathscr{F}(K_0\backslash G_0/K_0)$ onto $\mathscr{F}(K\backslash G/U)$ such that*

$$f^{\eta}(x) = f(x\sigma(x)^{-1}), \qquad x \in G_0 . \tag{5.3}$$

(ii) *There is an isomorphism $D \to D^{\eta}$ of $\mathbf{D}_R(K_0\backslash G_0) \otimes \mathbf{D}(G_0/K_0)$ onto $\mathbf{D}_R(K\backslash G)$ and of $\mathbf{Z}(G_0)$ onto $\mathbf{D}(G/U)$, such that*

$$(Df)^{\eta} = D^{\eta}f^{\eta} \tag{5.4}$$

for all $f \in C^{\infty}(K_0\backslash G_0/K_0)$.

Remark. From now on we shall identify $\mathscr{F}(K_0\backslash G_0/K_0)$ with $\mathscr{F}(K\backslash G/U)$, and write f instead of f^{η}, when no confusion is possible. One should notice, that if ω_0 and ω are the Cassimir operators on G_0 and G, respectively, then $(\omega_0)^\eta = \frac{1}{2}\omega$. Also if φ_λ is a spherical function on G_0 then for $H \in a_0$

$$(\varphi_\lambda)^\eta(\exp H) = \varphi_\lambda(\exp(2H)).$$

COROLLARY 5.3. *φ is a spherical function on G_0 if and only if*

$$\varphi \in C^\infty(K\backslash G/U), \qquad \varphi(e) = 1$$

and

$$D\varphi = \lambda_D \varphi \qquad \textit{for all} \quad D \in \mathbf{D}_R(K\backslash G). \tag{5.5}$$

Proof. This is an immediate consequence of Theorem 5.2 and Definition (2.12). Q.E.D.

We have thus proved Theorem 1.1.

COROLLARY 5.4. *If φ is a spherical function on G_0, then $\Phi \in C^\infty(U\backslash G/U)$, defined by*

$$\Phi(x) = \int_U \varphi(ux)\,du, \qquad x \in G, \tag{5.6}$$

is a spherical function on G.

Proof. Since $\mathbf{Z}(G_0)$ restricted to G_0/K_0 is contained in $\mathbf{D}(G_0/K_0)$, it follows by Theorem 5.2 (ii), that Φ is an eigenfunction of all $D \in \mathbf{D}(G/U)$. Clearly $\Phi(e) = 1$ and the corollary follows from Definition (2.12). Q.E.D.

This corollary defines using (2.13) a mapping $\lambda \to \Lambda$ from $(a_0)^*_{\mathbb{C}}(\text{mod } W_0)$ into $a^*_{\mathbb{C}}(\text{mod } W)$. In the next theorem we describe this mapping. Recall that $a = a_0 + a_k$, that we have embedded a_0^* in a^*, and that $\rho = 2(\rho_0 + \rho_k)$.

THEOREM 5.5. *For each $\lambda \in (a_0)^*_{\mathbb{C}}$ define $\Lambda \in a^*_{\mathbb{C}}$ by*

$$\Lambda = 2(\lambda - i\rho_k), \tag{5.7}$$

then

$$\Phi_\Lambda(x) = \int_U \varphi_\lambda(ux)\,du. \tag{5.8}$$

Proof. We shall first prove this theorem for the discrete subset F of λ's in $(a_0)^*_{\mathbb{C}}$ for which the spherical function φ_λ is a matrix coefficient of a finite-dimensional representation of G_0. The following properties of these representations follow from [16, Chap. III, Section 3]. If $\lambda \in F$, then λ is purely imaginary, so by conjugating by W_0 we can assume $\lambda \in F^+ = \{\lambda \in F \mid i\lambda \in \overline{a_0^{*+}}\}$. F^+ is given by

$$F^+ = \left\{\lambda \in (a_0)^*_{\mathbb{C}} \;\middle|\; \frac{\langle i\lambda - \rho_0\,,\, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Delta_0^+\right\}. \tag{5.9}$$

If $\lambda \in F^+$ then $\Lambda_0 = i\lambda - \rho_0$ is the dominant weight of the corresponding finite-dimensional class-one representation E_Λ of G_0.

Let $E^*_{\Lambda_0}$ be the contragridient representation to E_{Λ_0}. We can take $E^*_{\Lambda_0}$ to be $E_{-\Lambda_0}$, where we choose an ordering "opposite" of the one for E_{Λ_0}, i.e., we define $-a^+$ to be the positive Weyl chamber. Then $E = E_{\Lambda_0} \otimes E_{-\Lambda_0}$ is an irreducible representation of $G_0 \times G_0$, which is of class-one w.r.t. $K_0 \times K_0$, and has a fixed vector for $d(G_0)$.

The dominant weight of $g \times g$ corresponding to E is $(\Lambda_0, -\Lambda_0)$. The spherical function on $G_0 \times G_0$ corresponding to E is

$$\varphi_\lambda(x)\,\varphi_{-\lambda}(y) = \varphi_\lambda(x)\,\varphi_\lambda(y^{-1}) = \int_{K_0} \varphi_\lambda(xky^{-1})\,dk$$

$$= \int_{K_0 \times K_0} \varphi_\lambda(xk_1, yk_2)\,dk_1\,dk_2$$

for $(x, y) \in G_0 \times G_0$. (The first equality follows from [29, vol II, Proposition 6.2.2.1], in the last we have used (5.2).) Clearly $(x, y) \to \varphi_\lambda(x, y) = \varphi_\lambda(xy^{-1})$, contained in $C^\infty(K_0 \times K_0\backslash G_0 \times G_0/d(G_0))$, is a matrix coefficient of E. E extends to a holomorphic representation $E^{\mathbb{C}}$ of $G \times G$. Let E^η be the restriction of $E^{\mathbb{C}}$ to G (embedded as in Lemma 5.1). Now by analytic continuation we see that $(\varphi_\lambda)^\eta \in C^\infty(K\backslash G/U)$ is a matrix coefficient of E^η.

It is clear that E^η has a fixed vector for K and is of class-one w.r.t. U. Using (5.9) again we see that the corresponding spherical function is Φ_Λ, with $i\Lambda - \rho = (\Lambda_0, -\Lambda_0)$. Since a is embedded in $g \times g$ (see the proof of Lemma 5.1) as $\{(H, -H) \mid H \in a\}$, we get

$$i\Lambda(H) - \rho(H) = 2\Lambda_0(H) = 2i\lambda(H) - 2\rho_0(H),$$

from this follows that $\Lambda = 2(\lambda - i\rho_k)$. As we have seen that $(\varphi_\lambda)^\eta$ is a matrix coefficient of E^η, we finally get

$$\Phi_\Lambda(x) = \int_U \varphi_\lambda(ux)\,du.$$

So we have proved Theorem 5.5 for $\lambda \in F$.

Now to prove (5.7) for an arbitrary $\lambda \in (a_0)^*_{\mathbb{C}}$, define Φ_Λ by (5.8), then for each $D \in \mathbf{D}(G/U)$

$$D\varphi_\lambda = P_0(\lambda)\,\varphi_\lambda, \qquad D\Phi_\Lambda = P(\Lambda)\,\Phi_\Lambda,$$

where by the remark after (2.13) P_0 and P are polynomials. We have shown that if $\mu \in F$ then $P_0(\mu) = P(2(\mu - i\rho_k))$. Since this is a polynomial identity, it holds for all $\mu \in (a^0)^*_{\mathbb{C}}$. So we can conclude that Φ_Λ and $\Phi_{2(\lambda - i\rho_k)}$ have the same eigenvalues for each $D \in \mathbf{D}(G/U)$, hence they are equal. Q.E.D.

PROPOSITION 5.6. *The mapping* $\varphi_\lambda \to \Phi_{2(\lambda - i\rho_k)}$ *is injective, whenever the following property holds*

$$\{P\,|_{a_0} \mid P \in \mathbf{I}_W(a)\} = \mathbf{I}_{W_0}(a_0). \tag{5.10}$$

Proof. (5.10) is equivalent to the property that the restriction of $\mathbf{Z}(G_0)$ to G_0/K_0 is surjective onto $\mathbf{D}(G_0/K_0)$, see Helgason [14, p. 590]. This means, by Theorem 5.2, that the eigenvalues of φ_λ for all $D \in \mathbf{D}(G_0/K_0)$ is determined by the eigenvalues of $\Phi_{2(\lambda - i\rho_k)}$ for all $D \in \mathbf{Z}(G_0)$. Hence φ_λ is uniquely determined by $\Phi_{2(\lambda - i\rho_k)}$. Q.E.D.

PROPOSITION 5.7. (Helgason [14]). *Property* (5.10) *holds if* g_0 *does not contain any simple ideal which is a non-normal form of type* E_6, E_7, *or* E_8.

EXAMPLE. Let us compute the eigenvalue Λ_ω of $\Phi_{2(\lambda - i\rho_k)}$ for ω. We use (2.14):

$$\begin{aligned}\Lambda_\omega &= -(\langle 2(\lambda - i\rho_k), 2(\lambda - i\rho_k)\rangle + \langle \rho, \rho\rangle)\\ &= -(\tfrac{1}{2}\langle 2\lambda, 2\lambda\rangle_0 - \langle 2\rho_k, 2\rho_k\rangle + \langle 2\rho_0 + 2\rho_k, 2\rho_0 + 2\rho_k\rangle)\\ &= -2(\langle \lambda, \lambda\rangle_0 + \langle \rho_0, \rho_0\rangle_0).\end{aligned}$$

This is twice the eigenvalue of φ_λ for ω_0, in accordance with the fact that $(\omega_0)^\eta = \frac{1}{2}\omega$.

Now we indicate how Theorem 2.3, which characterizes the bounded spherical functions on G_0, can be proved by a reduction to the same theorem for the complex group G. We shall need the following lemma:

LEMMA 5.8. *Let* $\lambda \in (a_0)^*_{\mathbb{C}}$, *then*

$$\lambda \in a_0^* + iC_{\rho_0} \quad \textit{if and only if} \quad 2(\lambda - i\rho_k) \in a^* + iC_\rho.$$

Proof. Two simple consequences of Lemma 2.5 in [18] are

(i) If $\eta \in a_0^{*+}$, then $\eta \in C_{\rho_0}$ if and only if

$$\langle \eta - \rho_0, H\rangle_0 < 0 \qquad \text{for all } H \in a_0{}^+.$$

(ii) If $\eta \in a_0^*$ and there exists $H \in \overline{a_0{}^+}$ such that $\langle \eta - \rho_0, H\rangle_0 > 0$, then $\eta \notin C_{\rho_0}$.

It follows from (2.3) that $2(i\rho_0 - i\rho_k)$ is conjugate to $2i(\rho_0 + \rho_k) = i\rho$ under W. Therefore $2(iC_{\rho_0} - i\rho_k) \subset iC_\rho$, and thus $2(a_0^* + iC_{\rho_0} - i\rho_k) \subset a^* + iC_\rho$.

Now assume that $\lambda = \xi + i\eta$ and $\eta \notin C_{\rho_0}$, conjugating by W_0 we may take $-\eta \in \overline{a_0{}^+}$. By (i) there exists $H_0 \in a_0{}^+$ such that $\langle -\eta - \rho_0, H_0\rangle_0 > 0$ and thus $\langle 2(-\eta + \rho_k) - \rho, H_0\rangle = \langle 2(-\eta - \rho_0), H_0\rangle > 0$. Since $H_0 \in a_0{}^+ \subset \overline{a^+}$ we now conclude from (ii) that $-2(\eta - \rho_k) \notin C_\rho = -C_\rho$, and therefore that

$$2(\lambda + i\eta - i\rho_k) \notin a^* + iC_\rho. \qquad \text{Q.E.D.}$$

Now assume that $\lambda \in a_0^* + iC_{\rho_0}$, then the boundedness of φ_λ is an immediate consequence of the maximum principle for the entire function $\lambda \to \varphi_\lambda(g)$ and the fact that $|\varphi_{\xi+is\rho}(g)| \leqslant \varphi_{is\rho}(g) = 1$ for $\xi \in a_0^*$, $s \in W$, $g \in G_0$ (cf. [18, final remark]).

On the other hand assume that $\lambda \notin a_0^* + iC_{\rho_0}$, but that φ_λ is bounded. Then by (5.8) $\Phi_{2(\lambda - i\rho_k)}$ is also bounded, so by Theorem 2.3 for G, $2(\lambda - i\rho_k) \in a^* + iC_\rho$, but this contradicts Lemma 5.8.

COROLLARY 5.9. *Let $\lambda \in (a_0)_{\mathbb{C}}^*$, then φ_λ is bounded if and only if $\Phi_{2(\lambda - i\rho_k)}$ is bounded.*

6. APPLICATIONS TO THE SPHERICAL FOURIER TRANSFORM ON G_0/K_0

Let $f \in L^1_{loc}(K\backslash G/U)$ and $F \in L^1_{loc}(U\backslash G/U)$, define $M_0 f \in L^1_{loc}(U\backslash G/U)$ by

$$M_0 f(x) = \int_U f(ux)\, du, \qquad x \in G, \tag{6.1}$$

and $MF \in L^1_{loc}(K\backslash G/U)$ by

$$MF(x) = \int_K F(hx)\, dh, \qquad x \in G, \tag{6.2}$$

whenever well defined almost everywhere. If furthermore $F \cdot f \in L^1(G)$ then by (2.11) and Fubini's theorem

$$\int_G F(x) f(x)\, dx = \int_G F(x)\, M_0 f(x)\, dx = \int_{K\backslash G} MF(x) f(x)\, dx. \tag{6.3}$$

In particular we get by Theorems 5.5 and 2.3:

THEOREM 6.1. *Let $F \in L^1_{loc}(U\backslash G/U)$ and $\lambda \in (a_0)_{\mathbb{C}}^*$ be such that MF is well defined and $F\varphi_\lambda \in L^1(G)$. Then with $\Lambda = 2(\lambda - i\rho_k)$ we have*

$$(MF)^\sim(\lambda) = F^\wedge(\Lambda). \tag{6.4}$$

This holds in particular, if F has compact support, or if $F \in L^1$ and $\lambda \in a_0^* + iC_{\rho_0}$. We can now sudy the "real" spherical Fourier transform on G_0 by means of the "complex" spherical Fourier transform on G. As we saw in Section 3, this in turn can often be studied by means of the Euclidean Fourier transform.

We shall first discuss Theorem 2.2 (the Paley–Wiener theorem) and Theorem 2.4 (about the spherical L^1-Schwartz space) for G_0, and try to reduce them to the corresponding theorems for G.

LEMMA 6.2. (i) *If* $F \in C_c^\infty(U\backslash G/U)$ *has support in a ball of radius* R, *then* $MF \in C_c^\infty(K\backslash G/U)$ *has support in a ball of radius* $2^{1/2}R$.

(ii) *If* $F \in \mathscr{I}^1(G)$ *then* MF *belongs to*

$$\mathscr{I}_1^1(G_0) = \{f \in C^\infty(K\backslash G/U) \mid \forall D \in \mathbf{D}(G), \forall N \in \mathbb{Z}^+,\\ x \to (1 + |x|)^N Df \text{ belongs to } L^1(K\backslash G)\}.$$

Proof. (ii) Recall that $p = ik_0 + p_0$, that $ik_0 \perp p_0$, and that we have defined $|x|$, such that if $X \in p$ and $u_1, u_2 \in U$ then $|u_1 \exp X \cdot u_2| = \|X\|$. Now it is a simple consequence of the negative sectional curvature of G/U, that

$$|\exp Y| \leqslant |\exp X \exp Y| \qquad \text{for all } X \in p_0,\ Y \in ik_0.$$

Since $K = K_0 \exp(ik_0)$ we find

$$2^{-1/2} |y|_0 = |y| \leqslant |hy| \qquad \text{for all } y \in \exp(p_0),\ h \in K. \tag{6.5}$$

So assume that $F \in \mathscr{I}^1(G)$, then for all $D \in \mathbf{D}(G)$ and all $N \in \mathbb{Z}^+$ the function $F_{N,D}(x) = (1 + |x|)^N DF(x)$ is in $L^1(G)$. Therefore by (2.11) the function $y \to \int_K |F_{N,D}(hy)|\, dh$ is in $L^1(K\backslash G)$, and we find for $y \in \exp(p_0)$, using (6.5), that

$$|(1 + |y|)^N D(MF)(y)| = \left|(1 + |y|)^N \int_K DF(hy)\, dh\right|\\ \leqslant \int_K |(1 + |hy|)^N|\, DF(hy)|\, dh = \int_K |F_{N,D}(hy)|\, dh.$$

From which follows that $MF \in \mathscr{I}_1^1(G_0)$, and (ii) is proved.

(i) follows easily from (6.5). Q.E.D.

One can now ask when M is a bijection of $C_c^\infty(U\backslash G/U)$ onto $C_c^\infty(K\backslash G/U)$, or of $\mathscr{I}^1(G)$ onto $\mathscr{I}_1^1(G_0)$. For a normal real form G_0 we prove this, at the same time we prove the Paley–Wiener theorem (Theorem 2.2) and a slight variation of Theorem 2.4.

THEOREM 6.3. *Assume that* G_0 *is a normal real form. Then the spherical Fourier transform is a bijection*

(i) *of* $C_c^\infty(K_0\backslash G_0/K_0)$ *onto* $\mathscr{H}(a_0^*)$, *and*

(ii) *of* $\mathscr{I}_1^1(G_0)$ *onto* $\mathscr{H}_{\rho_0}(a_0^*)$.

Also $f \in C_c^\infty(K_0\backslash G_0/K_0)$ *has support in a ball of radius* R *if and only if* $f^\sim$ *is of exponential type* R. *Furthermore the mapping* M *is a bijection*

(iii) *of* $C_c^\infty(U\backslash G/U)$ *onto* $C_c^\infty(K_0\backslash G_0/K_0)$, *and*

(iv) *of* $\mathscr{I}^1(G)$ *onto* $\mathscr{I}_1^1(G_0)$.

Proof. It is easy to see that the spherical Fourier transform maps $C_c^\infty(K_0\backslash G_0/K_0)$ into $\mathscr{H}(a_0^*)$, and that $f^\sim$ has exponential type R, if f has support in a ball of radius R. (One uses that $f^\sim$ is the Euclidean Fourier transform on a_0 of the Radon transform $F_f(\exp H) = e^{\langle\rho,H\rangle}\int_N f(\exp H\cdot n)\,dn$, $H \in a_0$.) It is also fairly easy to show that if $f \in \mathscr{I}_1^1(G_0)$, then $f^\sim \in \mathscr{H}_{\rho_0}(a^*)$. (See Helgason [16, pp. 28–31], and notice that the partial integrations used to prove that

$$(\omega^n f)^\sim(\lambda) = -(\langle\lambda,\lambda\rangle_0 + \langle\rho_0,\rho_0\rangle_0)^n f^\sim(\lambda)$$

can be carried out, integrating $f^n\varphi_\lambda{}^n$ over $K\backslash G$, instead of $f\varphi_\lambda$ over G_0.)

If now ψ is a function on a^*, denote by $j(\psi)$ the function on a_0^* given by $\lambda \to \psi(2(\lambda - i\rho_k))$ (if $\rho_k \neq 0$ one assumes that ψ is holomorphic in a tube containing $a_0^* - 2i\rho_k$). If $\psi \in \mathscr{H}(a^*)$ is of exponential type R, then $j(\psi) \in \mathscr{H}(a_0^*)$ is of exponential type $2^{1/2}R$, (recall that $\|\lambda\|_0 = 2^{1/2}\|\lambda\|$ for all $\lambda \in a_0^*$). Also, if $\psi \in \mathscr{H}_\rho(a^*)$, then by Lemma 5.8 $j(\psi) \in \mathscr{H}_{\rho_0}(a_0^*)$. We have then the two following commuting diagrams:

$$\begin{array}{ccc} C_c^\infty(U\backslash G/U) & \xrightarrow{\text{“}\wedge\text{”}} & \mathscr{H}(a^*) \\ \big\downarrow M & & \big\downarrow j \\ C_c^\infty(K_0\backslash G_0/K_0) & \xrightarrow{\text{“}\sim\text{”}} & \mathscr{H}(a_0^*) \end{array} \tag{6.6}$$

and

$$\begin{array}{ccc} \mathscr{I}^1(G) & \xrightarrow{\text{“}\wedge\text{”}} & \mathscr{H}_\rho(a^*) \\ \big\downarrow M & & \big\downarrow j \\ \mathscr{I}_1^1(G_0) & \xrightarrow{\text{“}\sim\text{”}} & \mathscr{H}_{\rho_0}(a_0^*). \end{array} \tag{6.7}$$

Now $a_0 = a$, $W_0 = W$, and $\rho_k = 0$ since G_0 is a normal form. Therefore j is a bijection in (6.6) and (6.7). In particular j is surjective. From Section 3 we know that “$\wedge$” is bijective, and it is easy to see that “$\sim$” is injective (see Helgason [13, X, exercise E.1]). Thus we conclude that “$\sim$” is bijective, and that M is surjective. We know also that j is injective. So we conclude that M is bijective. The statement about the support follows from Lemma 6.2(i). Q.E.D.

Remark. One should notice that this proof of Theorem 6.3 does not depend on any knowledge of the c-function, the inversion formula or the Plancherel formula for G_0. It is a simple reduction to the complex case. If G_0 is not a normal form we cannot expect M to be injective, but under the following two conditions the rest of the theorem will hold for G_0, with the same proof.

$$j\colon \mathscr{H}(a^*) \to \mathscr{H}(a_0^*) \quad \text{is onto.} \tag{6.8}$$

$$j\colon \mathscr{H}_\rho(a^*) \to \mathscr{H}_{\rho_0}(a_0^*) \quad \text{is onto.} \tag{6.9}$$

We shall below make an attempt to prove (6.8); but only obtain the following partial results:

THEOREM 6.4. (i) *Let $\psi_0 \in \mathscr{H}(a_0^*)$ be of exponential type R, and assume that ψ_0 is radial (i.e. invariant under orthogonal transformations of a_0^*). Then there exists $\psi \in \mathscr{H}(a^*)$ of exponential type $2^{-1/2}R$, such that $j(\psi) = \psi_0$.*

(ii) *In particular if G_0/K_0 has rank one then $j: \mathscr{H}(a^*) \to \mathscr{H}(a_0^*)$ is onto.*

(iii) *Assume that G_0 satisfies* (5.10). *Let $\psi_0 \in \mathscr{H}(a_0^*)$. There exists an entire, W-invariant function ψ on $a_{\mathbb{C}}^*$, such that $j(\psi) = \psi_0$.*

Before we prove the theorem we derive a simple corollary and prove a lemma.

COROLLARY 6.5. *If G_0/K_0 has rank one then*

$$M: C_c^\infty(U\backslash G/U) \to C_c^\infty(K_0\backslash G_0/K_0) \text{ is onto.}$$

Proof. It is an immediate consequence of the proof of Theorem 6.3 and Theorem 6.4(ii). Q.E.D.

LEMMA 6.6. *Let $P_1, ..., P_n$ be algebraically independent polinomials on $(a_0)_{\mathbb{C}}^*$, such that $P_1, ..., P_n$ generate $\mathbf{I}(a_0^*)$, then*

$$\mathbb{C}^n = \{(P_1(x), ..., P_n(x)) \mid x \in \mathbb{C}^n\}.$$

Proof. By assumption the mapping $\Gamma: \mathbb{C}[P_1, ..., P_n] \to \mathbb{C}[X_1, ..., X_n]$, defined by $\Gamma(Q(P_1, ..., P_n)) \to Q(P_1(X), ..., P_n(X))$, is an injection. Let now $a \in \mathbb{C}^n$ determine the maximal ideal $\mathscr{I}_a = (P_1 - a_1) \cdots (P_n - a_n)\, \mathbb{C}[P_1, ..., P_n]$. By [13, X, Theorem 5.5(i) and Lemma 6.9], there exists a maximal ideal

$$\mathscr{I}_b \text{ in } \mathbb{C}[X_1, ..., X_n], \qquad \mathscr{I}_b = (X_1 - b_1) \cdots (X_n - b_n)\, \mathbb{C}[X_1, ..., X_n],$$

with $b \in \mathbb{C}^n$, such that

$$\mathscr{I}_b \cap \Gamma(\mathbb{C}[P_1, ..., P_n]) = \Gamma(\mathscr{I}_a).$$

In other words if $X_1 = b_1, ..., X_n = b_n$, then $P(X_1) = a_1, ..., P(X_n) = a_n$. Q.E.D.

Proof of Theorem 6.4 (iii). Let $n = \dim a$ and $m = \dim a_0$.

Let $P_1, ..., P_n$ be homogeneous, algebraically independent generators of $\mathbf{I}(a^*)$, the W-invariant polynomials on a^*. Let "$-$" denote restriction to a_0^*. We can assume by (5.10) that $\bar{P}_1, ..., \bar{P}_m$ are algebraically independent and generate $\mathbf{I}(a_0^*)$.

Choose a basis and write with $a_\nu \in \mathbb{C}$:

$$\psi_0(\lambda) = \sum_\nu a_\nu \lambda^\nu = \sum_{N \in \mathbb{Z}^+} \sum_{|\nu| = N} a_\nu \lambda^\nu, \qquad \lambda \in a_0^* \tag{6.10}$$

where $\nu \in (\mathbb{Z}^+)^m$, $\lambda^\nu = \lambda_1^{\nu_1} \cdots \lambda_m^{\nu_m}$ and $|\nu| = \nu_1 + \cdots + \nu_m$. Clearly $Q_N = \sum_{|\nu|=N} a_\nu \lambda^\nu$ is a W_0-invariant homogeneous polynomial. Let $\gamma_1, ..., \gamma_m$ be the degrees of $P_1, ..., P_m$. Define $\| \mu \| = \sum_{i=1}^m \mu_i \gamma_i$ for $\mu \in (\mathbb{Z}^+)^m$. Now we can write

$$Q_N = \sum_{\|\mu\|=N} b_\mu \bar{P}^\mu, \tag{6.11}$$

where $b_\mu \in \mathbb{C}$ and $\bar{P}^\mu = \bar{P}_1^{\mu_1} \cdots \bar{P}_m^{\mu_m}$. Combining (6.10) and (6.11) we get

$$\psi_0(\lambda) = \sum_{N \in \mathbb{Z}^+} \sum_{\|\mu\|=N} b_\mu \bar{P}^\mu(\lambda), \qquad \lambda \in a_0^*. \tag{6.12}$$

Now Lemma 6.6 and the absolute convergence of (6.10) imply the absolute convergence of

$$\Psi_0(t) = \sum_{N \in \mathbb{Z}^+} \sum_{\|\mu\|=N} b_\mu t^\mu, \tag{6.13}$$

for any value of the variable $t \in \mathbb{C}^n$.

We then write for $i = 1, ..., m$ and $\lambda \in (a_0)_{\mathbb{C}}^*$

$$P_i(2(\lambda - i\rho_k)) = 2^{\gamma_i} \bar{P}_i(\lambda) + R_i(\lambda);$$

where $R_i \in \mathbf{I}(a_0^*)$ and $\text{degree}(R_i) < \text{degree}(P_i)$. An easy induction shows that we can find $Q_1, ..., Q_m \in \mathbf{I}(a^*)$ such that $j(Q_i) = \bar{P}_i$, i.e.,

$$Q_i(2(\lambda - i\rho_k)) = P_i(\lambda) \qquad \text{for all} \quad \lambda \in (a_0)_{\mathbb{C}}^*. \tag{6.14}$$

The function

$$\psi(\Lambda) = \Psi_0(Q_1(\Lambda), ..., Q_m(\Lambda))$$

now satisfies Theorem 6.4 (iii). Q.E.D.

Remark. Let the notation be as in the above proof. For each Λ in $a_{\mathbb{C}}^*$ let λ in $(a_0)_{\mathbb{C}}^*$ be such that

$$P_i(\lambda) = Q_i(\Lambda), \qquad i = 1, ..., m. \tag{6.15}$$

Now in order to prove condition (6.8), it would be enough to prove that there exist constants $C_1 > 0$, $C_2 \in \mathbb{R}$ such that

$$\| \operatorname{Im} \lambda \|_0 \leqslant C_1 \| \operatorname{Im} \Lambda \| + C_2. \tag{6.16}$$

If we also want the exponential type of ψ to be $2^{-1/2}R$, if ψ_0 is of exponential type R (cf. proof of Theorem 6.3), we must have $C_1 = 2^{-1/2}$.

Proof of Theorem 6.4 (i) *and* (ii). We use the notation from the proof above We can assume that $\bar{P}_1(\lambda) = \langle \lambda, \lambda \rangle_0$, then we can take $Q_1(\Lambda) = \frac{1}{2}\langle \Lambda, \Lambda \rangle + 2\langle \rho_k, \rho_k \rangle$ in order for (6.14) to be satisfied. Since ψ_0 is radial, only $\bar{P}_1$ will occu in (6.12). So for given Λ in $a_{\mathbb{C}}^*$ we just have to find $\lambda \in (a_0)_{\mathbb{C}}^*$, such that (6.15) holds for $i = 1$ and (6.16) holds with $C_1 = 2^{-1/2}$. Let $\Lambda = \xi + i\eta$ and $\lambda = \xi_0 + i\eta_0$. We seek λ, such that

$$\begin{aligned}\langle \lambda, \lambda \rangle_0 &= 2(\| \xi_0 \|^2 - \| \eta_0 \|^2 + 2i\langle \xi_0 , \eta_0 \rangle) \\ &= \tfrac{1}{2}\langle \Lambda, \Lambda \rangle + 2\langle \rho_k , \rho_k \rangle \\ &= \tfrac{1}{2}(\| \xi \|^2 - \| \eta \|^2 + 2i\langle \xi, \eta \rangle) + 2\langle \rho_k , \rho_k \rangle,\end{aligned} \tag{6.17}$$

and

$$2^{1/2} \| \eta_0 \|_0 = 2 \| \eta_0 \| \leqslant \| \eta \| + C.$$

Choose an orthonormal basis of a^*, such that a_0^* is spanned by the first m basis vectors. Write $\lambda = (\lambda_1 ,..., \lambda_m , 0,..., 0)$ and $\Lambda = (\Lambda_1 ,..., \Lambda_n)$ accordingly. We try to solve (6.17) with λ of the form $\lambda = (\xi_0 + i\eta_0 , 0,..., 0)$, $\xi_0 , \eta_0 \in \mathbb{R}$. Since (6.17) is not changed by applying an orthogonal transformation to Λ, we can assume $\Lambda_3 = \Lambda_4 = \cdots = \Lambda_n = 0$. We have then reduced the problem to the case of $\dim a = 2$, $\dim a_0 = 1$, and we have with $\Lambda_i = \xi_i + i\eta_i$, $i = 1, 2$, that

$$4(\xi_0{}^2 - \eta_0{}^2 + 2i\xi_0\eta_0) = s^2 - t^2 + 2istc_0 + 4\langle \rho_k , \rho_k \rangle, \tag{6.18}$$

where $s = (\xi_1{}^2 + \xi_2{}^2)^{1/2}$, $t = (\eta_1{}^2 + \eta_2{}^2)^{1/2}$ and $c_0 = (st)^{-1}(\xi_1\eta_1 + \xi_2\eta_2)$. From which it easily follows that $4\eta_0{}^2 \leqslant \eta_1{}^2 + \eta_2{}^2$, which is exactly what we want.

Now since $\psi_0 \in \mathscr{H}(a_0^*)$ if of exponential type R we have

$$\sup_{\Lambda \in a_{\mathbb{C}}^*} | e^{-2^{-1/2}R\|\eta\|}\psi(\Lambda)| \leqslant \sup_{\lambda \in (a_0)_{\mathbb{C}}^*} | e^{-R\|\eta_0\|_0}\psi_0(\lambda)| < +\infty.$$

Therefore ψ has exponential type $2^{-1/2}R$. We just need to show that ψ is rapidly decreasing on a^*: but for $\Lambda \in a^*$ we have by (6.18) that $\| \lambda \|_0^2 = \frac{1}{2} \| \Lambda \|^2 + 2 \| \rho_k \|^2$, and therefore for all $N \in \mathbb{Z}^+$,

$$\begin{aligned}\sup_{\Lambda \in a^*}(1 + \| \Lambda \|)^N | \psi(\Lambda)| &\leqslant \sup_{\lambda \in a_0^*}(1 + 4 \| \rho_k \|^2 + 2 \| \lambda \|_0^2)^N | \psi_0(\lambda)| \\ &< \infty. \qquad \text{Q.E.D.}\end{aligned}$$

Example. The Gauss-kernel on G_0/K_0. The technique used in the proof of Theorem 6.4 can be used to "explicit" computations of the inverse spherical Fourier transform in certain cases.

$$\text{Let for } t > 0,\ \psi_0{}^t(\lambda) = e^{-(t/2)\|\lambda\|_0^2}, \qquad \lambda \in a_0^*.$$

Let g_t be defined such that $g_t{}^{\sim} = \psi_0{}^t$. Then $t \to g_t$ is a semigroup under convolution of positive definite functions on G_0. We want to compute g_t. First define

$$\psi^t(\Lambda) = e^{-(t/2)(1/2\langle \Lambda,\Lambda \rangle + 2\langle \rho_k,\rho_k \rangle)}, \qquad \text{for} \quad \Lambda \in a_{\mathbb{C}}^*, \tag{6.19}$$

then $j(\psi^t) = \psi_0{}^t$. Extend ψ^t to a U invariant holomorphic function on $\mathfrak{p}_{\mathbb{C}}^*$ by the same formula (6.19) with $\Lambda \in \mathfrak{p}_{\mathbb{C}}^*$. By formula (3.4) we find the inverse spherical Fourier transform G_t of ψ^t to satisfy

$$\psi^t = (G_t)^\wedge = \mathscr{F}_1(2^{(1/2)l}\pi'(\rho)\, G_t J^{1/2}).$$

From this we deduce that ($d = \dim \mathfrak{p} = \dim \mathfrak{g}_0$)

$$G_t(\exp X) = 2^{-(1/2)l}\pi'(\rho)^{-1} \left(\frac{t}{2}\right)^{-d/2} e^{-t\|\rho_k\|^2} J(X)^{-1/2} e^{-(1/t)\|X\|^2}, \qquad X \in \mathfrak{p},$$

or

$$G_t(x) = (2^{1/2})^{-(l+d)}\, \pi'(\rho)^{-1}\, t^{-d/2} e^{-t\|\rho_k\|^2} e^{-(1/t)|x|^2} \Phi_0(x), \qquad x \in G.$$

Finally using Theorem 6.1 we get

$$g_t(x) = (MG_t)(x) = \int_K G_t(hx)\, dh. \tag{6.20}$$

Remark. Theorem 6.1 can be used to obtain some a priori estimates for the Plancherel measure for G_0/K_0 : We choose bounded nondifferentiable functions F of compact support in $L^1(U\backslash G/U)$, whose spherical Fourier transform $F^\wedge$ does not behave better than the inverse of a polynomial on $\mathfrak{a}^* + iC_\rho$. (Use (3.4).) The Plancherel measure $d\mu$ for G_0/K_0 has to integrate $j(F^\wedge)^2$. So we can conclude that $d\mu$ is a "tempered" measure. (This is indeed the case by the Gindikin–Karpelevic formula for $c(\lambda)$.)

7. Spherical Fourier Analysis on a Normal Real Form

The aim of this section is to convince the reader that spherical Fourier analysis on a normal form G_0 is almost as easy as on a complex group. So we assume that G_0 is a normal form. We have already, in Section 6, given simple proofs of the Paley–Wiener therem and the corresponding theorem for L^1-Schwartz-functions.

THEOREM 7.1. *There exists an $\epsilon > 0$, such that if $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\|\operatorname{Im} \Lambda\| < \epsilon$ and $N \in \mathbb{Z}^+$ then $x \to (1 + |x|)^N \Phi_\Lambda(x)$ is integrable over K.*

Beginning of proof. By (3.2) it follows easily that

$$|\Phi_\Lambda(\exp X)| \leqslant J(X)^{-1/2} e^{\epsilon\|X\|} = \Phi_0(\exp X)\, e^{\epsilon|\exp X|}. \tag{7.1}$$

Also $h \to \Phi_0(h)\, e^{\epsilon|h|}$ is a K_0 biinvariant function on K, so by (2.8) we have to prove

$$\int_K \Phi_0(h) e^{\epsilon|h|}\, dh = \int_{i\mathfrak{k}} e^{\epsilon\|T\|} J(T)^{-1/2}\, J_1(T)\, dT < +\infty, \tag{7.2}$$

where J_1 is the "J-function" corresponding to K. J and J_1 are given by simple expressions involving the corresponding root structures, and Theorem 7.1 can be proved, by just studying these root structures. The details are left to the next section.

COROLLARY 7.2. *Let $\epsilon > 0$ be as in the theorem, and $\Omega_\epsilon = \{\lambda \in (a_0)^*_{\mathbf{C}} \mid \|\operatorname{Im} \lambda\|_0 < (2^{-1/2})\,\epsilon\}$. Define*

$$\gamma(\lambda) = \int_K \Phi_{2\lambda}(h)\, dh \qquad \text{for} \quad \lambda \in \Omega_\epsilon . \tag{7.3}$$

Then γ is holomorphic in Ω_ϵ.

Before stating the next theorem, we just remark that

$$\pi_0(\lambda)^2 = \prod_{\alpha\in\Delta_0^+} \langle\alpha, H_\lambda{}^0\rangle^2 = \prod_{\alpha\in\Delta^+} \langle\alpha, 2H_\lambda\rangle^2 = \pi'(2\lambda)^2 = \pi(2\lambda).$$

So in particular $\pi_0(\rho_0)^2 = \pi(\rho)$.

THEOREM 7.3. (The Plancherel theorem and the Inversion formula.) *Let $f \in C_c^\infty(K_0\backslash G_0/K_0)$, then with $\gamma_0(\lambda) = 2^{(1/2)\dim a}\pi_0(\rho_0)^{-2}\, \pi_0(\lambda)^2\, \gamma(\lambda)$*

$$f(x) = \int_{a_0^{*+}} f^{\sim}(\lambda)\, \varphi_\lambda(x)\, \gamma_0(\lambda)\, d\lambda, \qquad \text{for all} \quad x \in G_0\,, \tag{7.4}$$

$f \to f^\sim$ extends to an isometry of $L^2(K_0\backslash G_0/K_0)$ onto $L^2(a_0^{*+}, \gamma_0(\lambda)\, d\lambda)$.

Proof. By Theorem 6.3 (iii) there is a unique $F \in C_c^\infty(U\backslash G/U)$ such that $f = MF$. Using Theorem 6.1, we find that $F^\wedge(2\lambda) = f^\sim(\lambda)$ for all $\lambda \in a_0^*$. So we can use the inversion formula for the complex spherical Fourier transform (3.3) to obtain (recall that $a^* = a_0^*$, but that the measure on a^* is $2^{-(1/2)\dim a}$ times the measure on a_0^*):

$$\begin{aligned} f(e) = MF(e) &= \int_K F(h)\, dh \\ &= \int_K \pi(\rho)^{-1} \int_{a^{*+}} F^\wedge(\Lambda)\, \Phi_\Lambda(h)\, \pi(\Lambda)\, d\Lambda\, dh \\ &= 2^{(1/2)\dim a}\pi_0(\rho_0)^{-2} \int_{a_0^{*+}} f^{\sim}(\lambda)\, \gamma(\lambda)\, \pi_0(\lambda)^2\, d\lambda. \end{aligned} \tag{7.5}$$

This proves (7.4) for $x = e$. Now the rest of the corollary is fairly standard. (Briefly: $\forall x_0 \in G$, $f^{x_0}(x) = \int_K f(xkx_0)\, dk$, $(f^{x_0})^\sim(\lambda) = f^\sim(\lambda)\, \varphi_\lambda(x_0)$; $f^*(x) = \bar{f}(x^{-1})$, $(f^* * f)^\sim(\lambda) = |f^\sim(\lambda)|^2$, $\|f\|_2^2 = f^* * f(e) = \int_{a_0^{*+}} |f^\sim(\lambda)|^2\, \gamma_0(\lambda)\, d\lambda$.)
Q.E.D.

COROLLARY 7.4. *For each $\lambda \in \Omega_\epsilon$, we have*

$$\gamma(\lambda)\, \varphi_\lambda(x) = \int_K \Phi_{2\lambda}(hx)\, dh.$$

Proof. We need the following fairly simple estimate for Φ_Λ: For each compact subset $B \subset G$, there exists $C > 0$ such that

$$|\Phi_\Lambda(yx)| \leq C\, |\Phi_\Lambda(y)| \qquad \text{for all } x \in B.$$

(This can be obtained from (2.13), see [9].) Therefore by Theorem 7.1, $x \to \int_K \Phi_{2\lambda}(hx)\, dh$ is well defined and continuous. Now proceeding as in (7.5) we find

$$f(x) = 2^{(1/2)\dim a}\pi_0(\rho_0)^{-2} \int_{a_0^{*+}} f^{\sim}(\lambda)\, \pi_0(\lambda)^2 \int_K \Phi_{2\lambda}(hx)\, dh\, d\lambda,$$

for all $x \in G$ and $f \in C_c^\infty(K\backslash G/U)$. Comparing with (7.4) the corollary follows, since $f^{\sim}$ runs through the whole space $\mathscr{H}(a_0^*)$. Q.E.D.

Remark. Using Theorem 2.1 we conclude that

$$\begin{aligned} |c(\lambda)|^{-2} &= \gamma_0(\lambda) = 2^{(1/2)\dim a}\pi_0(\rho)^{-2}\, \pi_0(\lambda)^2\, \gamma(\lambda), \quad \text{or} \\ |\pi_0(\lambda)\, c(\lambda)|^{-2} &= 2^{(1/2)\dim a}\pi_0(\rho_0)^{-2}\, \gamma(\lambda). \end{aligned} \tag{7.6}$$

Thus $\gamma(\lambda)$ is, up to a constant, the inverse square of the function $b(\lambda)$, introduced by Harish–Chandra in [9]. One should be able to prove (7.6) directly from formula (7.3).

For the normal forms we can also reduce the L^2-Schwartz-space theorem (Theorem 2.1 (i)) to the complex case: Define

$$\mathscr{I}_1^2(G_0) = \{f \in C^\infty(K\backslash G/U) \mid \forall D \in \mathbf{D}(G), \forall N \in \mathbb{Z}^+, \sup_{x\in G}(1 + |x|)^N Df \cdot \varphi_0^{-1} < +\infty\}. \tag{7.7}$$

THEOREM 7.5. (i) *The spherical Fourier transform is a bijection of $\mathscr{I}_1^2(G_0)$ onto $\mathscr{S}_{W_0}(a_0^*)$.*

(ii) *M is a bijection of $\mathscr{I}^2(G)$ onto $\mathscr{I}_1^2(G_0)$.*

Proof. We proceed exactly as in the proof of Theorem 6.4. We first prove that M maps $\mathscr{I}^2(G)$ into $\mathscr{I}_1^2(G_0)$. So assume $F \in F \in \mathscr{I}^2(G)$ (see Definition (2.15)). Let $N \in \mathbb{Z}^+$ and $D \in \mathbf{D}(G)$, then $(1 + |x|)^N\, |DF(x)| \leq c\Phi_0(x)$. So by Theorem

7.1 the following exchange of differentiation and integration is allowed, we also use (6.5) and assume $x \in \exp(\mathfrak{p}_0)$

$$
\begin{aligned}
&(1 + |x|)^N \, |D(MF)(x)| \\
&\quad = (1 + |x|)^N \left| \int_K DF(hx)\, dh \right| \\
&\quad \leqslant \int_K (1 + |hx|)^N \, |DF(hx)| \, dh \leqslant c \int_K \Phi_0(hx)\, dh = c \cdot \gamma(0) \cdot \varphi_0(x).
\end{aligned}
$$

Which is what we have to prove, to see that $MF \in \mathscr{I}_1^2(G_0)$.

Next we observe that it is fairly simple to prove that $\mathscr{I}_1^2(G_0)$ is mapped into $\mathscr{S}_{W_0}(a_0^*)$, see [10, pp. 592–596]. We then look at the commutative diagram

$$
\begin{array}{ccc}
\mathscr{I}^2(G) & \xrightarrow{\text{``}\wedge\text{''}} & \mathscr{S}_W(a^*) \\
\Big\downarrow{\scriptstyle M} & & \Big\downarrow{\scriptstyle j} \\
\mathscr{I}_1^2(G_0) & \xrightarrow{\text{``}\sim\text{''}} & \mathscr{S}_{W_0}(a_0^*).
\end{array}
$$

Clearly j is bijective since $a_0 = a$ and $W_0 = W$, and the Theorem follows. Q.E.D.

Remark. We should explain the relationship between the various definitions of the L^p-Schwartz-spaces, $p = 1$ or 2:

$\mathscr{I}^p$ defined in (2.18) using $\mathbf{D}(G_0)$.

$\mathscr{I}_0^p$ defined as in (3.10), using only the Laplace–Baltrami operator.

$\mathscr{I}_1^p$ defined as in Lemma 6.2 or as in (7.7), using $\mathbf{D}(G)$, considering $\mathscr{I}_1^p \subset C^\infty(K\backslash G/U)$.

The spaces are clearly contained in the space

$$
\begin{aligned}
\mathscr{I}_2^p = \{f \in C^\infty(K_0\backslash G_0/K_0) \mid \forall N, M \in \mathbb{Z}^+,\\
x \to (1 + |x|)^N \omega^M f(x) \text{ is in } L^p(G_0)\}.
\end{aligned}
$$

Which in turn is easily seen to be mapped into $\mathscr{S}_{W_0}(a_0^*)$ if $p = 2$, and into $\mathscr{H}_{\rho_0}(a_0^*)$ if $p = 1$. So by Theorems 7.5 and 6.3 we have $\mathscr{I}_1^2 = \mathscr{I}_0^2 = \mathscr{I}_2^2$, and $\mathscr{I}_1^1 = \mathscr{I}_2^1$.

Using Theorems 2.1 and 2.4, we get $\mathscr{I}^p = \mathscr{I}_2^p$. It is not quite clear how to prove a priori, that $\mathscr{I}^p = \mathscr{I}_2^p$. However, often $\mathscr{I}_1^p$ or, even $\mathscr{I}_2^p$, is enough to have for applications. In this connection we should notice that if definition $\mathscr{I}_2^p$ is used, then our method boils down to a very simple proof of Theorems 2.1 and 2.4 for the normal forms, since we can use Theorem 3.3 for G instead of Theorem 2.1.

8. The Proof of Theorem 7.1

Again in this section g_0 is a normal real form of g, $g_0 = k_0 + p_0$, $u = k_0 + ip_0$ and $g = u + iu$. We choose the maximal Abelian subalgebra a of $p = iu$ differently: First choose a_1 to be a maximal Abelian subalgebra in ik_0, then extend it to a maximal Abelian subalgebra a of p, such that $a = a_1 + a_2$, where $a_2 \subset p_0$. Δ is the roots of (g, a), and Δ_1 is the roots of (k, a_1). Notice that all roots have multiplicity 2. We choose a Weyl chamber a_1^+ in a_1. Let Δ_1^+ be the corresponding set of positive roots. Define $\rho_1 = \frac{1}{2} \sum_{\beta \in \Delta_1^+} 2\beta = \sum_{\beta \in \Delta_1^+} \beta$. We now choose a Weyl chamber a^+ in a, such that $H_{\rho_1} \in \overline{a^+}$. Δ^+ is the corresponding set of positive roots. $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} 2\alpha = \sum_{\alpha \in \Delta^+} \alpha$. The main ingredient in the proof of Theorem 7.1 is the following result:

Theorem 8.1. *If $H \in \overline{a_1^+} \cap \overline{a^+}$ and $H \neq 0$ then*

$$\langle \rho - 2\rho_1 , H \rangle > 0.$$

Proof. Let τ be the involution of g, introduced in Section 5, then τ is $+1$ on a_1 and -1 on a_2. Clearly τ induces an automorphism $\alpha \to \alpha^\tau$ of Δ onto itself. It follows now easily that every root $\beta \in \Delta_1$ is the restriction $\beta = \alpha|_{a_1}$ of an $\alpha \in \Delta$. Also if $\beta = \alpha|_{a_1} \in \Delta_1$, then $\alpha \in \Delta^+$ if and only if $\beta \in \Delta_1^+$. And we have the following splitting of Δ^+:

$$\begin{aligned} Q_1 &= \{\alpha \in \Delta^+ \mid \beta = \alpha|_{a_1} \in \Delta_1^+, \alpha = \alpha^\tau\}, \\ Q_2 &= \{\alpha \in \Delta^+ \mid \beta = \alpha|_{a_1} \in \Delta_1^+, \alpha \neq \alpha^\tau\}, \\ Q_3 &= \{\alpha \in \Delta^+ \mid \alpha|_{a_1} \notin \Delta_1\}. \end{aligned}$$

If $\alpha \in Q_1$, $\beta = \alpha|_{a_1}$, then $g^\alpha = g^\beta$. If $\alpha \in Q_2$, $\beta = \alpha|_{a_1}$, then $g^\beta = (g^\alpha + g^{\alpha\tau}) \cap k$. And we get

$$\rho - 2\rho_1 = \sum_{\alpha \in Q_3} \alpha - \sum_{\alpha \in Q_1} \alpha. \tag{8.1}$$

Remark. $b = ia_1 + a_2$ is a fundamental Cartan subalgebra of g_0, and the roots corresponding to Q_1 are the compact roots.

The rest of the proof now goes by classification. It is simple but a little tedious. We refer to [2] or [26] for the models of the root structures. For each type of root system A_n, B_n, C_n, D_n, G_2, F_4, E_8, E_7, and E_6, we list g, k, a, Δ, a_1, Δ_1, and either a^+, or Δ_1^+ and Δ^+ $(= Q_1 \cup Q_2 \cup Q_3)$. It is then a fairly simple matter to check that $H_{\rho_1} \in \overline{a^+}$, and to find

$$\rho - 2\rho_1 = \sum_{\alpha \in Q_3} \alpha - \sum_{\alpha \in Q_1} \alpha.$$

In all the cases we have written $\rho - 2\rho_1$ as a sum over l positive, orthogonal roots, whose restrictions to a_1 are also orthogonal ($l = \dim a_1$). From this the property of the theorem follows.

$\{e_1, \ldots, e_n\}$ shall denote an orthonormal basis in Euclidean n-space. The dual space is identified with $\mathbb{R}^n$, such that $\langle t, e_i \rangle = t_i$ for $i = 1, \ldots, n$. In several cases we have to distinguish between odd and even n. One of the cases is then carried on in parenthesis.

1. A_n . $n \geqslant 1$. $g = sl(n+1, \mathbb{C})$, $k = so(n+1, \mathbb{C})$, $n + 1 = 2m$ (or $2m+1$).

$$a = \{e_1 + \cdots + e_{n+1}\}^{\perp} \subset \mathbb{R}^{n+1},$$
$$\Delta = \{e_i - e_j \mid i \neq j,\ i, j = 1, \ldots, n+1\},$$
$$a_1 = (e_1 + e_2)^{\perp} \cap \cdots \cap (e_{2m-1} + e_{2m})^{\perp},\ (\cdots \cap e_{n+1}^{\perp}).$$

Define $f_1 = e_1 - e_2, \ldots, f_m = e_{2m-1} - e_{2m}$.

$$\Delta_1 = \{\tfrac{1}{2}(\pm f_i \pm f_j) \mid i \neq j,\ i, j = 1, \ldots, m\},$$
$$(\cdots \cup \{\pm \tfrac{1}{2} f_i \mid i = 1, \ldots, m\}).$$
$$a^+ = \{t \mid t_1 > -t_2 > t_3 > \cdots > -t_{2m}\} \cap a,$$
$$(\cdots > -t_{2m} > t_{2m+1}\} \cap a).$$
$$Q_1 = \varnothing, \qquad Q_3 = \{f_1, \ldots, f_m\}, \qquad \rho - 2\rho_1 = f_1 + \cdots + f_m .$$

2. B_n . $n \geqslant 2$. $g = so(2n+1, \mathbb{C})$, $k = so(n, \mathbb{C}) \times so(n+1, \mathbb{C})$.

$$a = a_1 = \mathbb{R}^n,\ n = 2m\ (n = 2m+1).$$
$$\Delta = \{\pm e_i \pm e_j \mid i \neq j,\ i, j = 1, \ldots, n\} \cup \{\pm e_i \mid i = 1, \ldots, n\},$$
$$\Delta_1 = \{\pm e_i \pm e_j \mid i \neq j,\ i, j \leqslant m \text{ or } i, j \geqslant m+1\}$$
$$\cup \{\pm e_i \mid i = m+1, \ldots, n\}\ (\cdots \cup \{\pm e_i \mid i = 1, \ldots, m\}).$$
$$a^+ = \left\{ t \mid t_{m+1} > t_1 > t_{m+2} > t_2 > \cdots > t_m > \left\{ \begin{matrix} 0 \\ ((t_{2m+1} > 0)) \end{matrix} \right\} \right\}.$$
$$Q_2 = \varnothing, \qquad \rho - 2\rho_1 = e_1 + \cdots + e_n .$$

3. C_n . $n \geqslant 3$. $g = sp(n, \mathbb{C})$, $k = gl(n, \mathbb{C})$.

$$a = a_1 = \mathbb{R}_n, \qquad n = 2m(n = 2m+1).$$
$$\Delta = \{\pm e_i \pm e_j \mid i \neq j,\ i, j = 1, \ldots, n\} \cup \{\pm 2e_i \mid i = 1, \ldots, n\}.$$

Define

$$f_1 = e_1 + e_n,\ f_n = e_1 - e_n,$$
$$f_2 = e_2 + e_{n+1}, \ldots, \left\{ \begin{matrix} f_{m+1} = e_m - e_{m+1} \\ ((f_{m+2} = e_m - e_{m+2},\ f_{m+1} = 2e_{m+1})) \end{matrix} \right\}.$$

$$\Delta_1 = \{e_i - e_j \mid i \neq j,\, i, j = 1,\ldots, n\}.$$

$$a^+ = \left\{t_1 > -t_n > t_2 > -t_{n-1} > \cdots > \begin{Bmatrix} -t_{m+1} \\ (t_{m+1}) \end{Bmatrix} > 0\right\}.$$

$$Q_2 = \varnothing, \qquad \rho - 2\rho_1 = 2e_1 + \cdots + 2e_m + \begin{Bmatrix} 0 \\ (2e_{m+1}) \end{Bmatrix} = f_1 + \cdots + f_n .$$

4. D_n . $n \geqslant 4$. $g = so(2n, \mathbb{C})$, $k = so(n, \mathbb{C}) \times so(n, \mathbb{C})$.

$$a = \mathbb{R}^n, \qquad n = 2m\,(\text{or } n = 2m + 1).$$
$$\Delta = \{\pm e_i \pm e_j \mid i \neq j,\, i, j = 1,\ldots, n\}.$$

Define $f_1 = e_1 + e_{m+1}$, $f_{m+1} = e_1 - e_{m+1}$, $f_2 = e_2 + e_{m+2}$,..., $f_{2m} = e_m - e_{2m}$. If $n = 2m$ then $a = a_1$ (if $n = 2m + 1$ then $a_1 = e_n{}^{\perp}$).

$$\Delta_1 = \{\pm e_i \pm e_j \mid i \neq j;\, i, j \leqslant m \text{ or } m < i, j \leqslant 2m\}\ (\cdots \cup \{\pm e_i \mid i \neq n\}).$$

$$a^+ = \left\{t_1 > t_{m+1} > t_2 > t_{m+2} > \cdots > \begin{Bmatrix} t_m \\ (t_{2m}) \end{Bmatrix} > | t_n |\right\}.$$

$$\rho - 2\rho_1 = 2e_1 + \cdots + 2e_m = f_1 + \cdots + f_n .$$

5. G_2 . $k = sl(2, \mathbb{C}) \times sl(2, \mathbb{C})$.

$a = a_1 = \mathbb{R}^2$. Simple roots α, β.

$\Delta = \{\pm\beta,\, \pm(\beta + \alpha),\, \pm(\beta + 2\alpha),\, \pm(\beta + 3\alpha),\, \pm(2\beta + 3\alpha),\, \pm\alpha\}$.

$f_1 = \beta, f_2 = \beta + 2$ orthogonal roots.

$\Delta_1 = \{\pm\alpha,\, \pm(2\beta + 3\alpha)\}$.

$\Delta_1{}^+ = \{\alpha, 2\beta + 3\alpha\}$, $\Delta^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha, 2\beta + 3\alpha, \alpha\}$.

$$\rho - 2\rho_1 = 2(\alpha + \beta) = f_1 + f_2 .$$

6. F_4 . $k = sp(3, \mathbb{C}) \times sl(2, \mathbb{C})$.

$a = a_1 = \mathbb{R}^4$.

$$\Delta = \{\pm e_i \mid i = 1,\ldots, 4\} \cup \{\pm e_i \pm e_j \mid i \neq j,\, i, j = 1,\ldots, 4\} \cup \{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

$f_1 = \frac{1}{2}(e_1 + e_2)$, $f_2 = \frac{1}{2}(e_1 - e_2)$ and $f_3 = \frac{1}{2}(e_3 + e_4)$.

$$\Delta_1 = \{\pm(e_3 - e_4)\} \cup \{\pm f_i \pm f_j \mid i \neq j,\, i, j = 1, 2, 3\} \cup \{\pm 2f_1,\, \pm 2f_2,\, \pm 2f_3\}.$$

$$a^+ = \{t \mid t_1 > t_3 > t_2 > t_4 > 0 \text{ and } t_1 > t_3 + t_4 + t_5\}.$$

$$\rho - 2\rho_1 = e_1 + e_2 + e_3 + e_4 .$$

7. E_8. $k = so(16, \mathbb{C})$. $a = a_1 = \mathbb{R}^8$.

$$\Delta = \{\pm e_i \pm e_j \mid i \neq j, i, j = 1,..., 8\} \cup \left\{\tfrac{1}{2} \sum_{i=1}^{8} (-1)^{m_i} e_i \;\middle|\; \sum_{i=1}^{8} m_i \equiv 0 \ (\mathrm{mod}\ 2)\right\}.$$

$\Delta_1 = \{\pm e_i \pm e_j \mid i \neq j, i, j = 1,..., 8\}$.

$\Delta_1^- = \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant 8\}$, $\rho_1 = 14e_1 + 12e_2 + \cdots + 2e_7$.

$$Q_2 = \varnothing, \qquad Q_3 = \left\{\alpha = \sum_{i=1}^{8} (-1)^{m_i} e_i \;\middle|\; \sum_{i=1}^{8} m_i \equiv 0 \ (\mathrm{mod}\ 2);\ \langle \alpha, \rho_1 \rangle > 0 \text{ or } (\langle \alpha, \rho_1 \rangle = 0 \text{ and } m_1 = 0)\right\}.$$

$$\rho - 2\rho_1 = 4e_1 = f_1 + f_2 + \cdots + f_8 .$$

Where $f_1, ..., f_8$ are all in Q_3 and given by the following combinations of signs:

	$(-1)^{m_1}$	·	·	·	·	·	·	$(-1)^{m_8}$
f_1, f_2	$+$	$+$	$+$	$+$	$\pm$	$\pm$	$\pm$	$\pm$
f_3, f_4	$+$	$+$	$-$	$-$	$\pm$	$\pm$	$\mp$	$\mp$
f_5, f_6	$+$	$-$	$+$	$-$	$\pm$	$\mp$	$\pm$	$\mp$
f_7, f_8	$+$	$-$	$-$	$+$	$\pm$	$\mp$	$\mp$	$\pm$

8. E_7. $k = sl(8, \mathbb{C})$, $\quad a_1 = a = (e_1 + \cdots + e_8)^\perp$.

$$\Delta = \{\alpha \in \Delta_{E_8} \mid \alpha^\perp(e_1 + \cdots + e_8)\} = \Delta_1 \cup (\pm Q_3),$$

$$\Delta_1 = \{e_i - e_j \mid i \neq j, i, j = 1,..., 8\},$$

$$\pm Q_3 = \left\{\sum_{i=1}^{8} (-1)^{m_i} e_i \mid m_i \in \{0, 1\}, \sum m_i = 4\right\}.$$

$$\rho_1 = 7e_1 + 5e_2 + 3e_3 + e_4 - e_5 - 3e_6 - 5e_7 - 7e_8 .$$

$$Q_3 = \{\alpha \in \pm Q_3 \mid \langle \alpha, \rho_1 \rangle > 0 \text{ or } (\langle \alpha, \rho_1 \rangle = 0 \text{ and } \langle \alpha, e_1 \rangle > 0)\}.$$

$$\rho - 2\rho_1 = 4e_1 - \tfrac{1}{2}(e_1 + \cdots + e_8) = f_2 + \cdots + f_8 ,$$

where $f_2, ..., f_8$ are given as under E_8.

9. E_6. $k = sp(4, \mathbb{C})$. $a = (e_7 + e_8)^\perp \cap (e_6 - e_7)^\perp$.

$$\Delta = \{\pm e_i \pm e_j \mid i \neq j, i, j = 1, 2,..., 5\} \cup \left\{\tfrac{1}{2} \sum_{i=1}^{8} (-1)^{m_i} e_i \mid m_6 = m_7, m_7 + m_8 = 1, \sum_{i=1}^{8} m_i \equiv 0 \ (\mathrm{mod}\ 2)\right\}.$$

$a_1 = \{t \mid t_5 = t_6 = t_7 = t_8 = 0\}$.

Define $f_1 = \frac{1}{2}(e_1 + e_2)$, $f_2 = \frac{1}{2}(e_1 - e_2)$, $f_3 = \frac{1}{2}(e_3 + e_4)$, $f_4 = \frac{1}{2}(e_3 - e_4)$.

$$\Delta_1 = \{\pm f_i \pm f_j \mid i \neq j, i, j = 1,..., 4\} \cup \{\pm 2f_i \mid i = 1,..., 4\}.$$
$$\Delta_1{}^+ = \{f_i \pm f_j \mid i < j\} \cup \{2f_i \mid i = 1,..., 4\}.$$
$$\rho_1 = 8f_1 + 6f_2 + 4f_3 + 2f_4 = 7e_1 + e_2 + 3e_3 + e_4 .$$
$$\Delta^+ = \{\alpha \in \Delta \mid \langle \alpha, \rho_1 \rangle > 0 \text{ or } (\langle \alpha, \rho_1 \rangle = 0 \text{ and } \langle \alpha, e_2 \rangle > 0)\}.$$
$$\rho - 2\rho_1 = 2e_1 + 2e_2 = g_1 + g_2 + g_3 + g_4 ,$$

where $g_1 = e_1 + e_3$, $g_2 = e_2 + e_4$, $g_3 = e_1 - e_3$, $g_4 = e_2 - e_4$. Q.E.D.

COROLLARY 8.2 (of the proof). *There exists an orthogonal system of roots $f_1,..., f_l \in \Delta^+$, where $l = \dim a_1$, such that the restrictions of $f_1,..., f_l$ to a_1 generate a_1^* and*

$$\rho - 2\rho_1 = f_1 + \cdots + f_l . \tag{8.2}$$

If the root system is not of type B_n, C_n, or F_4, $f_1,..., f_l$ can be chosen strongly orthogonal.

In the following we sketch a proof of (8.2) without use of classification. Clearly (8.2) implies Theorem 8.1.

Proof of (8.2). Let β be any simple root in Δ^+, then

$$\langle \rho, \beta \rangle / \langle \beta, \beta \rangle = 1 \qquad \text{and} \qquad \langle 2\rho_1 , \beta \rangle / \langle \beta, \beta \rangle \in \mathbb{Z}^+,$$

thus

$$\langle \rho - 2\rho_1 , \beta \rangle / \langle \beta, \beta \rangle = \begin{cases} 1 \\ \text{or } \leqslant 0. \end{cases}$$

Assume that

$$\langle \rho - 2\rho_1 , \beta \rangle / \langle \beta, \beta \rangle \leqslant 0 \qquad \text{for all simple roots.} \tag{8.3}$$

Then $\langle \rho - 2\rho_1 , \alpha \rangle \leqslant 0$ for all $\alpha \in \Delta^+$. Now choose a simple root $\beta_1 \in \Delta^+$. Define $a^1 = \{H \mid \beta_1(H) = 0\}$ and $\Delta^1 = \{\alpha \in \Delta \mid \langle \alpha, \beta \rangle = 0\}$. Then Δ^1 is naturally identified with the root structure of the complex Lie sub-algebra $g^1 = \{X \in g \mid [X, X_{\beta_1} + X_{-\beta_1}] = 0\}$, for a suitable choice of root vectors X_{β_1}, $X_{-\beta_1}$. $g^1 \cap g_0 = g_0{}^1$ is a normal real form of g^1. $g_0{}^1 = g_0{}^1 \cap k_0 + g_0{}^1 \cap p_0 = k_0{}^1 + p_0{}^1$ is the Cartan decomposition of $g_0{}^1$. Let $k^1 = k_0{}^1 + ik_0{}^1$, $a_1{}^1 = k_0{}^1 \cap a^1$, and let $\Delta_1{}^1$ be the root structure of $(k^1, a_1{}^1)$. Δ^+ induces an ordering determining Δ^{1+} and Δ_1^{1+}. Let $\rho^1 = \sum_{\alpha \in \Delta^{1+}} \alpha$, $\rho_1{}^1 = \sum_{\alpha \in \Delta_1^{1+}} \alpha$.

Then by the method of Schmid [25, Section 7] one can see that $\langle \rho - 2\rho_1 , H \rangle = \langle \rho^1 - 2\rho_1{}^1, H \rangle$ for all $H \in a^1$. From Vogan [28] it follows that $H_{\rho_1{}^1} \in \overline{a^{1+}}$. It follows from the assumption (8.3) that $\langle \rho^1 - 2\rho_1{}^1, \beta \rangle = \langle \rho - 2\rho_1 , \beta \rangle \leqslant 0$ for all $\beta \in \Delta_1{}^+$. So we choose a simple root $\beta^2 \in \Delta_1{}^+$, etc After each step the dimension of $a_1{}^i$ is reduced by one. When $\dim a_1{}^i = 1$ we are left with $g^i = sl(2, \mathbb{R})$, but in that case, $\rho_1{}^i = 0$ and $\rho \neq 0$, which is a contradiction to (8.3).

Then there exists a simple root β_1 such that

$$\langle \rho - 2\rho_1 , \beta_1 \rangle / \langle \beta_1 , \beta_1 \rangle = 1.$$

This means that $\langle \rho - 2\rho_1 - \beta_1 , \beta_1 \rangle = 0$, such that $\rho - 2\rho_1 - \beta_1 = \rho^1 - 2\rho_1{}^1$. Again using the induction we find that $\rho - 2\rho_1 = \beta_1 + \cdots + \beta_l$, $l = \dim \mathfrak{a}_1$.
Q.E.D.

Proof of Theorem 7.1. We have to prove that there is an $\epsilon > 0$, such that $e^{\epsilon \| T \|} J(T)^{-1/2} J_1(T)$ is integrable over $i\mathfrak{k}$, (see (7.2)). Where $J_1(T) = \delta_1(T) \pi_1(H)^{-1}$, $\delta_1(T) = \prod_{\alpha \in \Delta_1^+} (\sinh\langle \alpha, T \rangle)^2$ and $\pi_1(T) = \prod_{\alpha \in \Delta_1^+} \langle \alpha, T \rangle^2$. Using polar coordinates (2.6) it is enough to prove that $e^{\epsilon \| T \|} J(H)^{-1/2} \delta_1(H)$ is integrable over $\mathfrak{a}_1{}^+$. Now it is very easy to see that there are constants c_1, $c_2 > 0$ and $d \in \mathbb{Z}^+$ such that $\delta_1(H) \leqslant c_1 e^{\langle 2\rho_1, H \rangle}$ for all $H \in \overline{\mathfrak{a}_1{}^+}$, and

$$J(H) \geqslant c_2 (1 + \| H \|)^{-2d} e^{\langle 2\rho, H \rangle} \qquad \text{for all} \quad H \in \overline{\mathfrak{a}^+}.$$

Therefore for $H \in \overline{\mathfrak{a}_1{}^+} \cap \overline{\mathfrak{a}^+}$ we have

$$J(H)^{-1/2} \delta_1(H) \leqslant c_1 c_2^{-1/2} (1 + \| H \|)^d e^{-\langle \rho - 2\rho_1, H \rangle}.$$

Theorem 8.1 then shows that we can choose $\epsilon > 0$, such that $e^{\epsilon \| H \|} J(H)^{-1/2} \delta_1(H)$ is integrable over $\overline{\mathfrak{a}_1{}^+ \cap \mathfrak{a}^+}$.

Now $\overline{\mathfrak{a}_1{}^+} = \bigcup_{s \in I} s(\overline{\mathfrak{a}_1{}^+} \cap \overline{\mathfrak{a}^+})$, where the union is over a subset I of W. This is clear if $\mathfrak{a}_1 = \mathfrak{a}$. If $\mathfrak{a}_1 \neq \mathfrak{a}$ it follows, since $\Delta^1 = \{\alpha |_{\mathfrak{a}_1} \mid \alpha \in \Delta\}$ is a root system on $\mathfrak{a}_1$. If $H \in \overline{\mathfrak{a}_1{}^+} \cap \overline{\mathfrak{a}^+}$ and $s \in I$, then $J(sH) = J(H)$, and by [9, p. 280], $\rho_1(sH) \leqslant \rho_1(H)$ since $H_{\rho_1} \in \overline{\mathfrak{a}^+}$. We then have

$$\delta_1(sH) \leqslant c_1 e^{\langle 2\rho_1, sH \rangle} \leqslant c_1 e^{\langle 2\rho_1, H \rangle},$$

which shows that $e^{\epsilon \| H \|} J(H)^{-1/2} \delta_1(H)$ is integrable over $s(\overline{\mathfrak{a}_1{}^+} \cap \overline{\mathfrak{a}^+})$. Q.E.D.

9. Applications to Representation Theory

Assume that φ_λ, $\lambda \in (\mathfrak{a}_0)_{\mathbb{C}}^*$, corresponds to a finite-dimensional representation E_0 of G_0. Let $E_{\mathbb{C}}$ be the corresponding holomorphic representation of G. Let E_U be the restriction to U of E. Let E_0^*, $E_{\mathbb{C}}^*$, and E_U^* be the same for the contragridient representation. Consider the representation $E = E_{\mathbb{C}} \otimes E_{\mathbb{C}}^*$ of $G \times G$. Then E restricted to $U \times U$ is just $E_U \otimes E_U^*$. Denote E^η the restriction of E to $G_1 = \{x, \sigma(x)) \mid x \in G\}$.

The spherical function on U, w.r.t. K_0 corresponding to E_U, is just the analytic continuation of φ_λ to U. The character of E_U is $\dim(E_U)$ times the spherical function on $U \times U$, w.r.t. $\mathrm{d}(U)$, corresponding to $E_U \otimes E_U^*$. The analytic continuation of this spherical function to G_1 is just $\Phi_{2(\lambda - i\rho_k)}$, the spherical function on G_1, w.r.t. U_1, corresponding to E^n. Thus we conclude that our mapping $\varphi_\lambda \to \Phi_{2(\lambda - i\rho_k)} = \int_U \varphi_\lambda(u\cdot)\, du$ is the analogue, for noncompact groups, of the process of associating the character to a finite dimensional representation, for compact groups.

Let G_0/H_0 be a pseudo-Riemannian symmetric space, and K_0 a maximal compact subgroup of G_0, as in Section 4. In general the Plancherel measure for G_0/H_0 (i.e., the decomposition of the regular representation on $L^2(G_0/H_0)$ into irreducibles) is not known explicitly. It is quite natural to think that the restriction $d\mu_1$ of the continuous part of the Plancherel measure to $G_0\hat{}(K_0)$ (the set of class-one representations), should be given by $d\mu_1(\lambda) = |c(\lambda)|^{-2}\, d\lambda$, where the c-function is associated with the set of generalized spherical functions given by

$$\begin{gathered} \varphi \in C^\infty(K_0\backslash G_0/H_0), \qquad \varphi(e) = 1 \\ D\varphi = \lambda_D\varphi, \qquad \text{for all} \quad D \in \mathbf{D}(G_0/H_0). \end{gathered} \tag{9.1}$$

This is indeed true for the special case of G/K (where G is complex semisimple, and K is the complexification of a maximal compact subgroup K_0 of a real form G_0):

The generalized spherical functions are by (5.5) just the set $\{\varphi_\lambda{}^\eta \mid \lambda \in (a_0)^*_{\mathbb{C}}\}$. So the associated c-function is just the c-function for G_0. And the restricted Plancherel measure $d\mu_1$ is determined by decomposing the U-fix-vectors in $L^2(G/K)$, i.e., $L^2(U\backslash G/K)$, according to the generalized spherical functions. But since $L^2(U\backslash G/K)$ is the "same" as $L^2(K_0\backslash G_0/K_0)$, we find (using Theorems 5.2 and 5.5) that $d\mu_1$ is supported on the set $\{2(\lambda - i\rho_k) \mid \lambda \in a_0^*\}$ and there it is given by $|c(\lambda)|^{-2}\, d\lambda$. One aspect of this is that $\Phi_{2(\lambda - i\rho_k)}$ has to be positive definite for $\lambda \in a_0^*$.

If $\rho_k \neq 0$ (i.e., G_0 is neither a normal real form nor a Steinberg normal form), then this part of the Plancherel measure is not related to the unitary class-one principal series induced from a minimal parabolic subgroup of G. In any case it must be related to the unitary class-one principal series induced from the parabolic subgroup $P_0{}^{\mathbb{C}} = M_0{}^{\mathbb{C}} A_0{}^{\mathbb{C}} N_0{}^{\mathbb{C}}$, where $N_0{}^{\mathbb{C}} = \exp(n_0 + in_0)$, $A_0{}^{\mathbb{C}} = \exp(a_0 + ia_0)$ and $M_0{}^{\mathbb{C}}$ is the centralizer of a_0 in G. If G_0 is not a normal real form, then $P_0{}^{\mathbb{C}}$ is not minimal, and we are dealing with degenerate principal series. Rossmann [24] has suggested that one uses the correspondingly defined parabolic subgroup of G_0, to treat analysis on the pseudo-Riemannian symmetric spaces G_0/H_0 in general.

10. An Example, the Case of $\mathbf{SL}(2, \mathbb{R})$

$g_0 = sl(2, \mathbb{R})$, $g = sl(2, \mathbb{C})$. For our purpose we may assume $G = \mathbf{SO}_0(3, 1)$ $(\approx \mathbf{SL}(2, \mathbb{C})/\{\pm 1\})$, $U = \mathbf{SO}(3)$, $G_0 = \mathbf{SO}_0(2, 1)(\approx \mathbf{SL}(2, \mathbb{R})/\{\pm 1\}$, $K_0 = \mathbf{SO}(2)$,

$K = \mathbf{SO}(2) \times \mathbf{SO}_0(1, 1)$ and $A = A_0 = \mathbf{SO}_0(1, 1)$. In order to describe how the embeddings are it is enough to give the matrix form of K and A:

$$K = \left\{ k_\theta \cdot h_t = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix} \middle| \ \theta \in [0, 2\pi],\ t \in \mathbb{R} \right\}$$

$$A = \left\{ a_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh y & \sinh y \\ 0 & 0 & \sinh y & \cosh y \end{pmatrix} \middle| \ y \in \mathbb{R} \right\}.$$

We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ such that $\langle \Lambda, \log(a_y) \rangle = \Lambda \cdot y$. Then $\Delta = \{\pm 1\}$, $\rho_0 = \frac{1}{2}$, and $\rho = 1$. $\pi_0(\rho_0) = 1/4$, $\pi_0(\lambda) = \frac{1}{2}\lambda$. The normalization of measures are such that

$$\int_{G_0} f(x)\, dx = \int_0^\infty f(a_y) 2 \sinh y \frac{dy}{\pi^{1/2}} = \int_{K/G} f^\eta(x)\, dx$$

for $f \in C_c(K_0 \backslash G_0 / K_0)$, and

$$\int_G F(x)\, dx = \int_0^\infty F(a_y)\, 4(\sinh y)^2 \left(\frac{2}{\pi}\right)^{1/2} dy$$

for $F \in C_c(U \backslash G / U)$, and

$$\int_K F(h)\, dh = 2^{-3/2} \int_{-\infty}^\infty F(h_t)\, dt$$

for $F \in C_c(K_0 \backslash K)$.

The spherical functions are given by

$$\Phi_\Lambda(a_y) = \frac{\sin \Lambda y}{\Lambda \sinh y},$$

$$\begin{aligned} \varphi_\lambda(a_y) &= P_{i\lambda - 1/2}(\cosh 2y) = \varphi_\lambda^{(0,-1/2)}(2y) = \varphi_{2\lambda}^{(0,0)}(y) \\ &= {}_2F_1(\tfrac{1}{2} + i\lambda, \tfrac{1}{2} - i\lambda, 1; -(\sinh y)^2), \end{aligned}$$

where P_ν is the Legendre function, $\varphi_\mu^{(\alpha,\beta)}$ the Jacobi-function (see for example [19]), and ${}_2F_1$ is the Hypergeometric function.

The Plancherel measure for G_0/K_0 is by Corollary 7.3 given by

$$\gamma_0(\lambda) \frac{d\lambda}{2\pi^{1/2}} = 2^{1/2} \cdot 16 \cdot \tfrac{1}{4}\lambda^2 \cdot \gamma(\lambda) \frac{d\lambda}{2\pi^{1/2}}, \quad \text{where}$$

$$\begin{aligned}\gamma(\lambda) = \int_K \Phi_{2\lambda}(h)\, dh &= 2^{-3/2} \int_{-\infty}^{\infty} \frac{\sin 2\lambda t}{2\lambda \sinh t}\, dt \\ &= 2^{-9/2} \frac{\pi \sinh \pi 2\lambda}{\cosh \pi 2\lambda + 1} \\ &= 2^{-9/2}\lambda^{-2}\, \pi\lambda \tanh \pi\lambda, \end{aligned} \tag{10.1}$$

and thus

$$\gamma_0(\lambda) \frac{d\lambda}{2\pi^{1/2}} = \pi\lambda \tanh \pi\lambda \frac{d\lambda}{2\pi^{1/2}}$$

which is in accordance with Theorem 2.1, since $|c(\lambda)|^{-2} = \pi\lambda \tanh \pi\lambda$.

The integral evaluated in (10.1) also occurs in Godement's determination of the Plancherel measure for $\mathbf{SL}(2, \mathbb{R})$ [7]. But the fact that the integrand is the spherical function on $\mathbf{SL}(2, \mathbb{C})$ does not seem to have been noticed. See also Harish-Chandra [12], where a similar integral is used to determine the Plancherel measure for G_0 itself.

The mappings M_0 and M take the following explicit form:

$$M_0 f(a_y) = 2^{-3/2}(\sinh y)^{-1} \int_{s=0}^{2y} f(a_s)(\cosh 2y - \cosh s)^{-1/2}\, d(\cosh s) \tag{10.2}$$

for $f \in L^\infty(K_0\backslash G_0/K_0)$, and

$$MF(a_{2y}) = 2^{-3/2} \int_{s=|y|}^{\infty} F(a_s)(\cosh^2 s - \cosh^2 y)^{-1/2}(\cosh s)^{-1}\, d(\cosh^2 s) \tag{10.3}$$

for $F \in L^1(U\backslash G/U)$.

Since (10.3) is essentially a Weyl fractional integral of order $\frac{1}{2}$, the result of Theorem 6.3, that M is a bijection of $C_c^\infty(U\backslash G/U)$ onto $C_c^\infty(K_0\backslash G_0/K_0)$, follows easily in this case (see [19, p. 153]).

Now formulas (1.2) and (1.4) take the following form (after a simple change of variables):

$$\begin{aligned}\Phi_{2\lambda}(a_y) &= \frac{\sin 2\lambda y}{2\lambda \sinh y} \\ &= \frac{2^{-3/2}}{\sinh y} \int_{s=0}^{2y} \varphi_\lambda(a_s)(\cosh 2y - \cosh s)^{-1/2}\, d \cosh s, \end{aligned} \tag{10.4}$$

$$\varphi_\lambda(a_y) = 2^{1/2}(\pi \tanh \pi\lambda)^{-1} \int_{s=|y|}^{\infty} \sin \lambda s(\cosh s - \cosh y)^{-1/2}\, ds. \tag{10.5}$$

Finally we state formula (6.20) for the Gauss-kernel on $\mathbf{SL}(2, \mathbb{R})$:

$$g^t(a_y) = 2^{-9/2}t^{-3/2} \int_{s=|y|}^{\infty} s e^{-(1/t)s^2}(\cosh s - \cosh y)^{-1/2}\, ds. \tag{10.6}$$

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